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CWI Tract 86

On Topological and Linear Equivalence of certain Function Spaces

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CHAPTER 0

Introduction and notation

§0.1. Introduction

For a Tychonov space X, $C(X)$ is the set of all real-valued continuous functions on X. The set $C(X)$ endowed with the topology of pointwise convergence will be denoted $C_p(X)$ (for more precise definitions see section 1.1). The spaces $C_p(X)$ are of interest to topologists and functional analysts for various reasons.

One can consider $C_p(X)$ as a topological ring (with the usual addition and multiplication of functions). In [40], J. Nagata proved the following

0.1.1 THEOREM: *Let X and Y be Tychonov spaces. The spaces $C_p(X)$ and $C_p(Y)$ are topologically isomorphic as topological rings if and only if X and Y are homeomorphic.*

In this theorem it is essential to consider topological isomorphisms. There are non-homeomorphic spaces X and Y such that the rings $C(X)$ and $C(Y)$ are algebraically isomorphic (see [25]). Once we have J. Nagata's result it is natural to consider $C_p(X)$ as a topological vector space (with the usual addition and scalar multiplication) or just as a topological space. In view of this we can state two general problems.

0.1.2 PROBLEM: *Let X and Y be Tychonov spaces and suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic or just homeomorphic. Which topological properties $\mathcal{P}$ satisfy: X has property $\mathcal{P}$ if and only Y has property $\mathcal{P}$?*

0.1.3 PROBLEM: *Let X and Y be Tychonov spaces. Under what conditions on X and Y are $C_p(X)$ and $C_p(Y)$ linearly homeomorphic or just homeomorphic?*

Many topologists have worked on both problems. We will mention a few results that

are in the same spirit as the ones that will be derived in this monograph. (for a survey of recently obtained results we refer to [2]).

For example, concerning problem 0.1.2 we have for linear homeomorphisms a positive answer for pseudocompactness, compactness, σ-compactness (Arhangelskiĭ [1]), and dimension (Pestov [44]). A negative answer can be obtained for local compactness, first countability, second countability, metrizability, weight and character (cf. example 2.4.10 in this monograph). A useful strategy is to find pairs $(\mathcal{P}, \mathcal{Q})$ of topological properties such that a Tychonov space X satisfies $\mathcal{P}$ if and only if $C_p(X)$ satisfies $\mathcal{Q}$. In this way it is proved that for density (Guthrie [29]) and cardinality (Arhangelskiĭ [2]) problem 0.1.2 has a positive answer. On the other hand there exist a compact space X and a non-compact space Y such that $C_p(X)$ and $C_p(Y)$ are homeomorphic (cf. chapter 3 in this monograph).

In this monograph we present our contributions to problems 0.1.2 and 0.1.3 and related problems. We do not restrict ourselves to the topology of pointwise convergence. We also consider other topologies on $C(X)$ (mainly the compact-open topology on $C(X)$) and on $C^*(X)$, the set of all bounded real-valued continuous functions. Our results depend strongly on the results obtained by Arhangelskiĭ in [1]. We discuss [1] in detail in section 1.2.

In chapter 1 we mainly develop tools that will be important in later chapters. However, we also present some new results. In section 1.5 we prove for normal first countable spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, that the set of accumulation points of X is countably compact if and only if the set of accumulation points of Y is countably compact. The first countability assumption is essential. This result is joint work with J. van Mill [5]. Furthermore we prove in this section for metric spaces X and Y such that there is a continuous linear surjection from $C_p(X)$ onto $C_p(Y)$, that Y is completely metrizable whenever X is. This result is joint work with J. Pelant [7], and answers a well-known research problem of Arhangelskiĭ.

In chapter 2 we deal with function spaces of locally compact spaces. We give a complete isomorphical classification of the function spaces $C_p(X)$ and $C_0(X)$ (as topological vector spaces) where X is a member of one of the following classes:

(a) compact zero-dimensional metric spaces (section 2.4)
(b) compact ordinals (section 2.5)
(c) σ-compact ordinals (section 2.6)
(d) separable metric zero-dimensional locally compact spaces (section 2.7).

The isomorphical classification of the function spaces $C_0(X)$, for X an element of class

(a) or (b) is an old result (cf. [10] and [34]). The first three sections contain preliminaries which are of particular importance in this chapter. They deal with ordinals (in particular the for us important notion of a prime component), scattered spaces and factorizing lemmas on function spaces.

After the results in chapter 2, it is natural to consider non-locally compact spaces. In chapter 3 we prove for non-locally compact countable metric spaces X and Y that $C_p(X)$ and $C_p(Y)$ are homeomorphic. This result is joint work with J. van Mill and J. Pelant [6]. It was later extended to non-discrete countable metric spaces (see [16] or [20]), by different techniques.

In chapter 4 we consider linear homeomorphisms between function spaces $C_p(X)$ for metric spaces X. A new tool is developed there, namely the notion of l_p-equivalent pair. This notion provides us with many properties for which problem 0.1.2 can be positively answered in the class of zero-dimensional separable metric spaces (sections 4.1 and 4.3). A complete isomorphical classification will be given for function spaces $C_p(X)$ (as topological vector spaces) of countable metric spaces X with scattered height less than or equal to ω (section 4.2). We indicate in section 4.4 that an isomorphical classification for function spaces $C_p(X)$ for all countable metric spaces X seems beyond reach. Finally in this chapter some results will be given concerning the compact-open topology (section 4.5) and concerning the set of bounded continuous real-valued functions (section 4.6). We construct locally compact countable metric spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, while $C_p^*(X)$ and $C_p^*(Y)$ are not linearly homeomorphic.

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§0.2. Notation

For all undefined notions and results on general topology without explicit reference we refer to [23] and [24].

Notation	*Meaning*
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
C	the Cantor set
$X \approx Y$	X and Y are homeomorphic
$X \sim Y$	X and Y are linearly homeomorphic
$\bar{A}$	the closure of A in X
$\operatorname{Int} A$	the interior of A in X
$\min A$	the minimum of A
$\max A$	the maximum of A
$\sup A$	the supremum of A
$\inf A$	the infimum of A
$B(x, \varepsilon)$	$\{y \in X : d(x,y) < \varepsilon\}$
$\operatorname{diam} A$	$\sup\{d(x,y) : x, y \in A\}$
$d(X)$	the density of X
$\lvert X \rvert$	the cardinality of X
$\mathcal{P}(X)$	the power set of X
$X \oplus Y$	the topological sum of the spaces X and Y
$\oplus_{i=1}^{\infty} X_i$	the topological sum of the spaces X_i
$f \circ g$	the composition of two functions f and g
$id_X, 1_X, 1$	the identity function of X
$\pi_i : \Pi_{i \in I} X_i \to X_i$	the projection on the i-th coordinate
χ_A	the characteristic function of A
$\lVert . \rVert$	the norm on a Banach space
$\operatorname{Ker} F$	the kernel of a linear function F
$\operatorname{span}\{v_1, \ldots, v_n\}$	the linear span of $\{v_1, \ldots, v_n\}$
$\operatorname{conv}\{v_1, \ldots, v_n\}$	the convex hull of $\{v_1, \ldots, v_n\}$
S^n	the unit sphere in $\mathbb{R}^{n+1}$
B^{n+1}	the unit ball in $\mathbb{R}^{n+1}$
$W(\alpha)$	the set of all ordinals smaller than α
$[1, \alpha]$	the space of ordinals $\{\beta : 1 \leq \beta \leq \alpha\}$ with the order topology
$[1, \alpha)$	the space of ordinals $\{\beta : 1 \leq \beta < \alpha\}$ with the order topology
ω	the first infinite ordinal
ω_1	the first uncountable ordinal
$\aleph_0$	the cardinality of ω

CHAPTER 1

Tools and first applications

All spaces considered in this chapter are Tychonov.

In this chapter we introduce function spaces endowed with several topologies. Our main interest will be the topology of pointwise convergence and the compact open topology. In section 2 we present important results of Arhangelskiĭ [1] which are among the main tools in this monograph. Section 3 deals with the topological dual of a function space endowed with the topology of pointwise convergence, and section 4 gives some more details about the results of section 2, when dealing with the topology of pointwise convergence. Finally in section 5 we give some first applications. We present topological properties which are preserved by ι_p, ι_0 or ι_b-equivalence (resp. ι_p^*, ι_0^* or ι_b^*-equivalence), and properties which are not preserved by ι_p, ι_0 or ι_b-equivalence (resp. ι_p^*, ι_0^* or ι_b^*-equivalence). For definitions of these notions see section 1.5.

§1.1. Topologies on function spaces

For a space X we define $C(X)$ to be the set of all real-valued continuous functions on X and $C^*(X)$ to be the set of all bounded real-valued continuous functions on X. $C(X)$ and $C^*(X)$ are vector spaces with the natural addition and scalar multiplication. For a covering $\mathcal{K}$ of X we define a topology on $C(X)$ by taking the family of all sets

$$<f, K, \varepsilon> = \{g \in C(X) : |f(x) - g(x)| < \varepsilon \text{ for every } x \in K\},$$

where $f \in C(X)$, $K \in \mathcal{K}$ and $\varepsilon > 0$, as a subbase. If $\mathcal{K}$ is a covering of X consisting of compacta, $C(X)$ endowed with this topology is easily seen to be a topological group, whence in this case it generally suffices to consider open sets $<0, K, \varepsilon>$, where $K \in \mathcal{K}$ and $\varepsilon > 0$.

A subset A of a space X is said to be ***bounded*** whenever for every $f \in C(X)$, $f(A)$ is

bounded in $\mathbb{R}$. Note that if A is a bounded subset of X, then $\bar{A}$ is also a bounded subset of X. If $\mathcal{K}$ consists of all finite, compact or bounded subsets of X, respectively, we denote $C(X)$ endowed with this topology by $C_p(X)$, $C_0(X)$ or $C_b(X)$, respectively. The topology on $C_p(X)$ or $C_0(X)$, respectively, is often called ***the topology of pointwise convergence or the compact-open topology***, respectively. We have that $C_p(X)$ and $C_0(X)$ are topological vector spaces.

For spaces X and Y, the notation $X \leq Y$ means that X and Y have the same underlying set and the topology on Y is finer than or equal to the topology on X. With this notation we have

$$C_p(X) \leq C_0(X) \leq C_b(X).$$

1.1.1 LEMMA: *Let X be a space and let $\mathcal{K}$ be a covering of X consisting of compacta. Let $A_1, \ldots, A_n \in \mathcal{K}$, $f_1, \ldots, f_n \in C(X)$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$. Then for every $f \in \bigcap_{i=1}^n \langle f_i, A_i, \varepsilon_i \rangle$, there is $\delta > 0$ such that $\langle f, \bigcup_{i=1}^n A_i, \delta \rangle \subset \bigcap_{i=1}^n \langle f_i, A_i, \varepsilon_i \rangle$.*

PROOF: For $i \leq n$, let $\gamma_i = \max\{ |f(x) - f_i(x)| : x \in A_i \}$. Then $\gamma_i < \varepsilon_i$. Let $\delta_i = \varepsilon_i - \gamma_i$, and $\delta = \min\{\delta_i : i \leq n\}$. We claim that this δ suffices. Let $g \in \langle f, \bigcup_{i=1}^n A_i, \delta \rangle$, $i \leq n$ and $x \in A_i$. Then

$$\begin{aligned} |g(x) - f_i(x)| &\leq |g(x) - f(x)| + |f(x) - f_i(x)| \\ &< \delta + \gamma_i \leq \varepsilon_i, \end{aligned}$$

hence $g \in \langle f_i, A_i, \varepsilon_i \rangle$. □

1.1.2 COROLLARY: *Let X be a space and let $\mathcal{K}$ be the covering of X consisting of all finite or compact subsets. Then*

(a) $\{\langle f, K, \varepsilon \rangle : f \in C(X), K \in \mathcal{K}, \text{ and } \varepsilon > 0\}$ *is a base for $C(X)$, and*

(b) $\{\langle f, K, \varepsilon \rangle : K \in \mathcal{K}, \text{ and } \varepsilon > 0\}$ *is a neighborhoodbase at f for $f \in C(X)$.* □

1.1.3 EXAMPLE: Lemma 1.1.1, and corollary 1.1.2, do not hold if $\mathcal{K}$ consists of all bounded subsets of X.

For example let $X = \mathbb{R}$. Consider the identity $id: \mathbb{R} \to \mathbb{R}$. Note that $[0, 1)$ is bounded in $\mathbb{R}$ and that $0 \in \langle id, [0, 1), 1 \rangle$. Suppose there are a bounded $A \subset \mathbb{R}$ and $\varepsilon > 0$ such that $\langle 0, A, \varepsilon \rangle \subset \langle id, [0, 1), 1 \rangle$. Let $f \in C(\mathbb{R})$ be defined by $f \equiv -\varepsilon/2$. Then $f \in \langle 0, A, \varepsilon \rangle$, but $|f(1 - \varepsilon/2) - (1 - \varepsilon/2)| = 1$, so $f \notin \langle id, [0, 1), 1 \rangle$. Contradiction.

For a covering $\mathcal{K}$ of X, a topology on $C(X)$ can also be generated by the subbase consisting of all sets

$$A(K, U) = \{g \in C(X) : g(K) \subset U\},$$

where $K \in \mathcal{K}$ and U is open in $\mathbb{R}$.

1.1.4 LEMMA: *Let X be a space and let $\mathcal{K}$ be the covering of all finite (resp. compact) subsets of X. Then*

$$\{A(K, U) : K \in \mathcal{K} \text{ and } U \text{ open in } \mathbb{R}\}$$

is a subbase for $C_p(X)$ (resp. $C_0(X)$).

PROOF: First let $K \in \mathcal{K}$, U open in $\mathbb{R}$ and $f \in A(K, U)$. For every $x \in K$, there is a neighborhood U_x of x and $\varepsilon_x > 0$ such that

$$f(U_x) \subset (f(x) - \frac{\varepsilon_x}{2}, f(x) + \frac{\varepsilon_x}{2}) \subset (f(x) - \varepsilon_x, f(x) + \varepsilon_x) \subset U.$$

Since K is compact, there are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^{n} U_{x_i}$. Let $\varepsilon = \min\{\varepsilon_{x_i} : i \leq n\}$. We claim that $<f, K, \varepsilon/2> \subset A(K, U)$. Indeed let $g \in <f, K, \varepsilon/2>$, and let $x \in K$. There is $i \leq n$ such that $x \in U_{x_i}$. Then obviously $|g(x) - f(x_i)| < \varepsilon_{x_i}$. So $g(x) \in U$. This implies $g \in A(K, U)$.

Second let $f \in C(X)$, $K \in \mathcal{K}$ and $\varepsilon > 0$. For every $x \in K$ let $U_x = f^{-1}((f(x) - \varepsilon/3, f(x) + \varepsilon/3))$ and let $C_x = \overline{U}_x \cap K$. Then each C_x is compact. Let $V_x = (f(x) - \varepsilon/2, f(x) + \varepsilon/2)$. There are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^{n} U_{x_i}$. We claim that $f \in \bigcap_{i=1}^{n} A(C_{x_i}, V_{x_i}) \subset <f, K, \varepsilon>$. Indeed, for every $i \leq n$, $f(C_{x_i}) \subset \overline{f(U_{x_i})} \subset V_{x_i}$. For $g \in \bigcap_{i=1}^{n} A(C_{x_i}, V_{x_i})$ and $x \in K$, there is $i \leq n$ such that $x \in C_{x_i}$, Then obviously $|f(x) - g(x)| < \varepsilon$. So $g \in <f, K, \varepsilon>$. □

1.1.5 EXAMPLE: Lemma 1.1.4 does not hold in case $\mathcal{K}$ consists of all bounded subsets. As in example 1.1.3 consider $<id, [0, 1), 1>$. Let $K_1, \ldots, K_n$ be bounded subsets of $\mathbb{R}$ and let $U_1, \ldots, U_n$ be open subsets of $\mathbb{R}$ such that $0 \in \bigcap_{i=1}^{n} A(K_i, U_i)$. There is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset \bigcap_{i=1}^{n} U_i$. Again let $f \equiv -\varepsilon/2$. Then $f \in \bigcap_{i=1}^{n} A(K_i, U_i)$ and as in example 1.1.3, $f \notin <id, [0, 1), 1>$.

When dealing with the topology of pointwise convergence or the compact-open topology we will use corollary 1.1.2 and lemma 1.1.4 without explicitly referring to it.

1.1.6 LEMMA: *$C_p(X)$ is a dense subspace of $\mathbb{R}^X$ with the product topology.*

PROOF: That $C_p(X)$ is a subspace of $\mathbb{R}^X$ is easily seen. Let $f \in \mathbb{R}^X$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$. We have to show that for

$$U = \{g \in \mathbb{R}^X : |f(x_i) - g(x_i)| < \varepsilon \text{ for every } i \leq n\}$$

we have $U \cap C_p(X) \neq \emptyset$. For $i \leq n$, let $f_i \in C(X)$ be a Urysohn function such that $f_i(x_i) = f(x_i)$ and $f_i(x_j) = 0$ for $j \leq n$ and $j \neq i$. Let $g = \Sigma_{i=1}^n f_i$. Then for every $i \leq n$, $g(x_i) = f(x_i)$, so $g \in U \cap C_p(X)$. □

We define $C_p^*(X)$, $C_0^*(X)$ and $C_b^*(X)$ similar to $C_p(X)$, $C_0(X)$ and $C_b(X)$ using $C^*(X)$ instead of $C(X)$. All the observation made above for $C(X)$ endowed with one of the defined topologies are also valid for $C^*(X)$ endowed with this topology.

On $C^*(X)$, we define ***the topology of uniform convergence*** by the metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\},$$

where $f, g \in C^*(X)$. We denote $C^*(X)$ endowed with this topology by $C_u^*(X)$. It is well-known that $C_u^*(X)$ is a Banach space ([47, Prop. 4.1.2]). It is easily seen that $C_0^*(X) \leq C_u^*(X)$. For a compact space X the topology of uniform convergence and the compact-open topology coincide ([24, Th. 4.2.17]).

All results in this section are well-known. The easy examples 1.1.3 and 1.1.5 were constructed by us. For more information about topologies on function spaces we refer to [24], [37] and [47].

§1.2. Linear functions between function spaces

In this section we present results which are of fundamental importance in this monograph. In particular we present results of Arhangelskiĭ [1] (corollaries 1.2.15 and 1.2.21).

Let X and Y be spaces and let ϕ: $C(X) \to C(Y)$ (resp. ϕ: $C^*(X) \to C^*(Y)$) be a linear function. For every $y \in Y$, the ***support of y in X with respect to*** ϕ is defined to be the set $\mathrm{supp}(y)$ of all $x \in X$ satisfying the condition that for every neighborhood U of x, there is $f \in C(X)$ (resp. $f \in C^*(X)$) such that $f(X \setminus U) \subset \{0\}$ and $\phi(f)(y) \neq 0$. Note that it suffices that the condition holds for arbitrarily small neighborhoods of x. For a subset A of Y we denote $\bigcup\{\mathrm{supp}(y) : y \in A\}$ by $\mathrm{supp}\, A$. Whenever ϕ is a linear bijection we can

consider the support of a point in Y with respect to ϕ and the support of a point in X with respect to ϕ^{-1}. It will always be clear from the context which support we mean. The following lemma is obvious, and is stated for reference purposes.

1.2.1 LEMMA: *Let X and Y be spaces and let $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) be a linear function. Let $y \in Y$. Then*

(a) $x \notin \operatorname{supp}(y)$ if and only if x has a neighborhood U such that for $f \in C(X)$ (resp. $f \in C^*(X)$) with $f(X \setminus U) \subset \{0\}$ we have $\phi(f)(y) = 0$, and

(b) $\operatorname{supp}(y)$ is closed in X. □

1.2.2 EXAMPLES: **(1)** Let X be a space. Define $\phi: C(X) \to C(X)$ by $\phi \equiv 0$. Obviously ϕ is linear. By lemma 1.2.1 (a), $\operatorname{supp}(x) = \emptyset$, for every $x \in X$.

(2) Let X be a space and let $\lambda \in \mathbb{R} \setminus \{0\}$. Define $\phi: C(X) \to C(X)$ by $\phi(f) = \lambda f$ for every $f \in C(X)$. Obviously ϕ is linear. We claim that for every $x \in X$, $\operatorname{supp}(x) = \{x\}$. First let U be any neighborhood of x. Let $f \in C(X)$ be a Urysohn function such that $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$. Then $\phi(f)(x) = \lambda \neq 0$, hence $x \in \operatorname{supp}(x)$. Second for $y \neq x$ let U be a neighborhood of y missing x. Then for $f \in C(X)$ with $f(X \setminus U) \subset \{0\}$, we have $\phi(f)(x) = 0$. By lemma 1.2.1 (a), $y \notin \operatorname{supp}(x)$.

(3) Let X be a space and let $x_0 \in X$ be fixed. Define $\phi: C(X) \to C(X)$ by $\phi(f) = f + f(x_0)$ for every $f \in C(X)$. Obviously ϕ is linear. We claim that for every $x \in X$, $\operatorname{supp}(x) = \{x, x_0\}$. Let U be any neighborhood of x. Let $V \subset U$ be a neighborhood of x such that if $x \neq x_0$, $x_0 \notin V$. Find a Urysohn function $f \in C(X)$ such that $f(x) = 1$ and $f(X \setminus V) \subset \{0\}$. Then $f(X \setminus U) \subset \{0\}$ and $\phi(f)(x) \neq 0$. Hence $x \in \operatorname{supp}(x)$. In a similar way one can prove that $x_0 \in \operatorname{supp}(x)$. As in (2) one can prove that for $y \notin \{x, x_0\}$, $y \notin \operatorname{supp}(x)$.

Each linear function above can also be defined from $C^*(X)$ to $C^*(Y)$.

The following definitions are due to Arhangelskiĭ [1]. Let X and Y be spaces. We say that a linear function $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) is ***effective*** if for every $f, g \in C(X)$ (resp. $f, g \in C^*(X)$) and $y \in Y$ such that f and g coincide on a neighborhood of $\operatorname{supp}(y)$, $\phi(f)(y) = \phi(g)(y)$. The linear function ϕ is of ***bounded type*** if ϕ is effective and for every $y \in Y$, $\operatorname{supp}(y)$ is bounded in X.

1.2.3 LEMMA: *Let X and Y be spaces and let $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) be a linear function which is not effective. Then there are $y \in Y$, a neighborhood U of $\operatorname{supp}(y)$ and $f \in C(X)$ (resp. $f \in C^*(X)$) such that $f(U) = \{0\}$ and $\phi(f)(y) \neq 0$.*

PROOF: Since ϕ is not effective, there are $y \in Y$, a neighborhood U of $\operatorname{supp}(y)$ and $f_1, f_2 \in C(X)$ (resp. $f_1, f_2 \in C^*(X)$) such that f_1 and f_2 coincide on U and $\phi(f_1)(y) \neq \phi(f_2)(y)$. Let $f = f_1 - f_2$. Then $f \in C(X)$ (resp. $f \in C^*(X)$). For $x \in U$ we have $f(x) = f_1(x) - f_2(x) = 0$ and by linearity of ϕ, $\phi(f)(y) = \phi(f_1)(y) - \phi(f_2)(y) \neq 0$. □

1.2.4 EXAMPLES: **(1)** The linear functions in example 1.2.2 have the property that for every $x \in X$ and for every $f, g \in C(X)$ such that f and g coincide on $\operatorname{supp}(x)$, $\phi(f)(x) = \phi(g)(x)$, hence they are effective.

(2) Let $X = [1, \omega_1)$ and let $Y = [1, \omega_1]$. Since every $f \in C(X)$ is eventually constant, i.e., there is $\alpha < \omega_1$ such that for each $\beta \geq \alpha$, $f(\alpha) = f(\beta)$ [24, example 3.1.27], f has a natural extension $\tilde{f} \in C(Y)$. The function $\phi: C(X) \to C(Y)$ defined by $\phi(f) = \tilde{f}$ is easily seen to be linear. We claim that ϕ is not effective. It is enough to show that $\operatorname{supp}(\omega_1) = \emptyset$, since in this situation any two functions in $C(X)$ coincide on a neighborhood of $\operatorname{supp}(\omega_1)$. Let $x \in X$. then $U = [1, x]$ is a neighborhood of x. Let $f \in C(X)$ be any mapping satisfying $f(X \setminus U) \subset \{0\}$. Then $\phi(f)(\omega_1) = 0$, hence by lemma 1.2.1 (a), $x \notin \operatorname{supp}(\omega_1)$.

Note that in this situation we have $C(X) = C^*(X)$ and $C(Y) = C^*(Y)$.

We will now give some general properties of effective linear functions between function spaces.

1.2.5 LEMMA: *Let X and Y be spaces and let $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) be an effective linear injection. Then $\overline{\operatorname{supp} Y} = X$.*

PROOF: Suppose there is $x \notin \overline{\operatorname{supp} Y}$. Then there is O open in X such that $\overline{\operatorname{supp} Y} \subset O$ and $x \notin \overline{O}$. Find a Urysohn function $f \in C^*(X)$ such that $f(x) = 1$ and $f(\overline{O}) \subset \{0\}$. Since O is a neighborhood of $\operatorname{supp} Y$ and ϕ is effective we then have $\phi(f)(Y) \subset \{0\}$. This implies $f \neq 0$ and $\phi(f) = 0$, contradicting the injectivity of ϕ. □

1.2.6 LEMMA: *Let X and Y be spaces and let $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) be an effective linear function. Then for $A \subset Y$ we have $\operatorname{supp} \overline{A} \subset \overline{\operatorname{supp} A}$.*

PROOF: Suppose there is $x \in \operatorname{supp} \overline{A} \setminus \overline{\operatorname{supp} A}$. Suppose $x \in \operatorname{supp}(y)$ for $y \in \overline{A}$. Find O open in X such that $x \subset O \subset \overline{O} \subset X \setminus \overline{\operatorname{supp} A}$. Since $x \in \operatorname{supp}(y)$ there is $f \in C(X)$ (resp. $f \in C^*(X)$) with $f(X \setminus O) \subset \{0\}$ and $\phi(f)(y) \neq 0$. Since $X \setminus \overline{O}$ is a neighborhood of $\operatorname{supp} A$ and $f \equiv 0$ on $X \setminus \overline{O}$, by effectiveness of ϕ, $\phi(f) \equiv 0$ on A. But this implies $\phi(f)(y) = 0$.

Contradiction. □

Let X and Y be spaces. A set-valued function $F: X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ such that for every $x \in X$, $F(x)$ is closed in Y is said to be ***Lower Semi Continuous*** (abbreviated LSC) whenever for every open $U \subset Y$ the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open in X. Consequently F is LSC if and only if for every closed $A \subset Y$ the set $\{x \in X : F(x) \subset A\}$ is closed in X. Furthermore F is said to be ***Upper Semi Continuous*** (abbreviated USC) whenever for every open $U \subset Y$ the set $\{x \in X : F(x) \subset U\}$ is open in X. Consequently F is USC if and only if for every closed $A \subset Y$ the set $\{x \in X : F(x) \cap A \neq \emptyset\}$ is closed in X.

If $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) is a linear function we can consider $\text{supp}: Y \to \mathcal{P}(X)$ as a set-valued function. We have

1.2.7 LEMMA: *Let X and Y be spaces and let $\phi: C(X) \to C(Y)$ (resp. $\phi: C^*(X) \to C^*(Y)$) be an effective linear function such that for each $y \in Y$, $\text{supp}(y) \neq \emptyset$. Then* supp *is LSC.*

PROOF: By lemma 1.2.1 (b), $\text{supp}(y)$ is closed in X for every $y \in Y$. Let U be an open subset of X. Put $O = \{y \in Y : \text{supp}(y) \cap U \neq \emptyset\}$, and let $y \in O$. Then there is $x \in \text{supp}(y) \cap U$. Let V be open in X such that $x \in V \subset \bar{V} \subset U$. Let $f \in C(X)$ (resp. $f \in C^*(X)$) be such that $f(X \setminus V) \subset \{0\}$ and $\phi(f)(y) \neq 0$. Let $W = \{z \in Y : \phi(f)(z) \neq 0\}$. Then W is an open neighborhood of y. We claim that $W \subset O$. Suppose there is $z \in W \setminus O$, i.e., $\phi(f)(z) \neq 0$ and $\text{supp}(z) \cap U = \emptyset$. Then $X \setminus \bar{V}$ is a neighborhood of $\text{supp}(z)$ and $f(X \setminus \bar{V}) \subset \{0\}$, so $\phi(f)(z) = 0$. Contradiction. So $W \subset O$ and hence the lemma is proved. □

REMARK: If the function ϕ in lemma 1.2.7 is surjective, then surely $\text{supp}(y) \neq \emptyset$ for every $y \in Y$. Indeed, if $\text{supp}(y) = \emptyset$ for some $y \in Y$, then let $f \in C^*(Y)$ be such that $f(y) \neq 0$. Choose $g \in C(X)$ (resp. $g \in C^*(X)$), such that $\phi(g) = f$. We have that $g \equiv 0$ on a neighborhood of $\text{supp}(y)$, so by effectiveness of ϕ, $f(y) = 0$. This gives a contradiction. We conclude that for any effective linear surjection, supp is LSC.

In section 2.4 we will give an example of an effective linear surjection $\phi: C(X) \to C(Y)$, such that supp is not USC.

1.2.8 PROPOSITION ([1]): *Let X and Y be spaces and $\phi: C(X) \to C(Y)$ a linear function of bounded type. Let A be a bounded subset of Y. Then* supp A *is bounded in X.*

PROOF: Suppose to the contrary that $\operatorname{supp} A$ is not bounded in X. Then there is $f \in C(X)$ such that $f(\operatorname{supp} A)$ is unbounded in $\mathbb{R}$, so there is $\{x_n : n \in \mathbb{N}\} \subset \operatorname{supp} A$ such that $\{f(x_n) : n \in \mathbb{N}\}$ is closed and discrete in $\mathbb{R}$. Hence we can find an open family $\{V_n : n \in \mathbb{N}\}$ in X such that $x_n \in V_n$ for each $n \in \mathbb{N}$ and $\{f(V_n) : n \in \mathbb{N}\}$ is a discrete family in $\mathbb{R}$. Then obviously $\{V_n : n \in \mathbb{N}\}$ is a discrete family in X.

By induction we construct a subset $\{y_k : k \in \mathbb{N}\}$ of A, a subfamily $\{U_k : k \in \mathbb{N}\}$ of $\{V_n : n \in \mathbb{N}\}$, and a subset $\{f_k : k \in \mathbb{N}\}$ of $C(X)$ such that

(1) $f_k(X \setminus U_k) \subset \{0\}$ for every $k \in \mathbb{N}$,
(2) for $i \neq j$ we have $U_i \neq U_j$,
(3) $\operatorname{supp}\{y_1, \ldots, y_{k-1}\} \cap \overline{U}_k = \emptyset$ for every $k > 1$, and
(4) $\phi(f_k)(y_k) = k + |h_k|$ for every $k \in \mathbb{N}$,
where $h_k = \Sigma_{i<k} \phi(f_i)(y_k)$ for $k > 1$ and $h_1 = 0$.

Let $y_1 \in A$ be such that $x_1 \in \operatorname{supp}(y_1)$. Let $U_1 = V_1$. Since U_1 is a neighborhood of x_1, and $x_1 \in \operatorname{supp}(y_1)$, there is $h \in C(X)$ such that $h(X \setminus U_1) \subset \{0\}$ and $\phi(h)(y_1) \neq 0$. Let

$$\lambda = \frac{1}{\phi(h)(y_1)}.$$

Let $f_1 = \lambda h$. Then $f_1 \in C(X)$ and $f_1(X \setminus U_1) \subset \{0\}$. Furthermore by linearity of ϕ, $\phi(f_1)(y_1) = \lambda \phi(h)(y_1) = 1$.

Let $k > 1$ and suppose we found $y_1, \ldots, y_{k-1}$, $U_1, \ldots, U_{k-1}$ and $f_1, \ldots, f_{k-1}$. Let $P_k = \operatorname{supp}\{y_1, \ldots, y_{k-1}\}$. Since ϕ is of bounded type, P_k is bounded in X. So there is $n \in \mathbb{N}$ such that $\overline{f(V_n)} \cap f(P_k) = \emptyset$ and hence $\overline{V}_n \cap P_k = \emptyset$. Since $x_n \in \operatorname{supp} A$, there is $y_k \in A$ such that $x_n \in \operatorname{supp}(y_k)$. Let $U_k = V_n$. Since U_k is a neighborhood of x_n and $x_n \in \operatorname{supp}(y_k)$, there is $h \in C(X)$ with $h(X \setminus U_k) \subset \{0\}$ and $\phi(h)(y_k) \neq 0$. Let

$$\lambda = \frac{k + |h_k|}{\phi(h)(y_k)}$$

and $f_k = \lambda h$. Then $f_k(X \setminus U_k) \subset \{0\}$ and by linearity of ϕ, $\phi(f_k)(y_k) = k + |h_k|$. To complete the inductive construction we observe that by (3) and the fact that $x_n \in \operatorname{supp}(y_k) \cap U_k$, $U_i \neq U_j$ for $i \neq j$.

Since $\{U_k : k \in \mathbb{N}\}$ is a subfamily of $\{V_n : n \in \mathbb{N}\}$ we have by (2) that $\{U_k : k \in \mathbb{N}\}$ is a discrete open family in X. Let $f = \Sigma_{i=1}^{\infty} f_i$. For $x \in X$ we have a neighborhood U_x which intersects at most one member of $\{U_k : k \in \mathbb{N}\}$. Then by (1) $f \mid U_x$ is a finite sum, hence $f \in C(X)$. For every $k \in \mathbb{N}$, let $g_k = \Sigma_{i=1}^{k} f_i$ and

$$W_k = X \setminus \bigcup_{j>k} \overline{f_j^{-1}(\mathbb{R} \setminus \{0\})}$$

By (1) we have for every $j \in \mathbb{N}$, $f_j^{-1}(\mathbb{R} \setminus \{0\}) \subset U_j$. So $\{f_j^{-1}(\mathbb{R} \setminus \{0\}) : j \in \mathbb{N}\}$ is a discrete family, hence W_k is open in X. By (3), we have for $j > k$, $\text{supp}(y_k) \cap \overline{U}_j = \emptyset$. This implies $\text{supp}(y_k) \cap \overline{f_j^{-1}(\mathbb{R} \setminus \{0\})} = \emptyset$. We conclude that $\text{supp}(y_k) \subset W_k$, so W_k is a neighborhood of $\text{supp}(y_k)$. For $j > k$, and $x \in W_k$, $f_j(x) = 0$, hence f and g_k coincide on W_k. Since ϕ is effective, we then have $\phi(f)(y_k) = \phi(g_k)(y_k)$. But $\phi(g_k)(y_k) = h_k + \phi(f_k)(y_k)$, so that by (4),

$$|\phi(g_k)(y_k)| \geq \phi(f_k)(y_k) - |h_k| = k + |h_k| - |h_k| = k.$$

We conclude that $|\phi(f)(y_k)| \geq k$ for every $k \in \mathbb{N}$. But this implies that A is not bounded in Y. Contradiction. □

1.2.9 COROLLARY: *Let X and Y be spaces and let $\phi: C(X) \to C(Y)$ be a linear injection of bounded type. If Y is pseudocompact, then X is pseudocompact.*

PROOF: If Y is pseudocompact it is bounded, hence by proposition 1.2.8, $\overline{\text{supp}\, Y}$ is bounded. By lemma 1.2.5, $\overline{\text{supp}\, Y} = X$. This implies that X is pseudocompact. □

We will now give two other applications of proposition 1.2.8. The first one will be a very important tool in chapter 4. The second one will be used in section 1.5.

1.2.10 LEMMA: *Let X and Y be normal spaces. Let K be compact and non-empty in Y and suppose $\{V_n : n \in \mathbb{N}\}$ is a decreasing base at K in Y. Let $\{A_s : s \in S\}$ be a locally finite family in X. Furthermore let $\phi: C(X) \to C(Y)$ be a linear function of bounded type. Then there are $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in S$ such that $(\text{supp}\, V_m) \cap \bigcup_{s \notin \{s_1, \ldots, s_m\}} A_s = \emptyset$.*

PROOF: If S is finite the lemma is obvious. Suppose the lemma is false for infinite S. Then there are distinct $s_i \in S$ ($i \in \mathbb{N}$) and points $x_i \in \text{supp}\, V_i \cap A_{s_i}$. Suppose $x_i \in \text{supp}\, y_i$ with $y_i \in V_i$. Since $\{A_{s_i} : i \in \mathbb{N}\}$ is locally finite, $\{x_i : i \in \mathbb{N}\}$ is infinite. Let $L = \{y_i : i \in \mathbb{N}\} \cup K$ and let $\mathcal{U}$ be an open cover of L. Then there are $U_1, \ldots, U_n$ in $\mathcal{U}$ such that $K \subset \bigcup_{i=1}^n U_i$. Since $\{V_n : n \in \mathbb{N}\}$ is a base at K in Y, there is $m \in \mathbb{N}$ such that $V_m \subset \bigcup_{i=1}^n U_i$. So $K \cup \{y_i : i \geq m\} \subset \bigcup_{i=1}^n U_i$. We conclude that L is compact.

By proposition 1.2.8, $\text{supp}\, L$ is bounded. It follows that $\{x_i : i \in \mathbb{N}\}$ is also bounded. However since $\{A_{s_i} : i \in \mathbb{N}\}$ is locally finite, $\{x_i : i \in \mathbb{N}\}$ is a closed and discrete set. Contradiction. □

Let X and Y be metric spaces and $\phi: C(X) \to C(Y)$ a linear surjection of bounded type. For $U \subset X$, let $T_U = \{y \in Y : \text{supp}(y) \cap U \neq \emptyset\}$. For a family $\mathcal{U}$ of subsets of X, let $T_{\mathcal{U}} = \{T_U : U \in \mathcal{U}\}$.

1.2.11 LEMMA: *If $\mathcal{U}$ is a locally finite open cover of X, then $T_{\mathcal{U}}$ is a locally finite open cover of Y.*

PROOF: By the remark following lemma 1.2.7, we have for each $y \in Y$, $\text{supp}(y) \neq \emptyset$. So $T_{\mathcal{U}}$ covers Y. Furthermore by lemma 1.2.7, supp is LSC, so for $U \in \mathcal{U}$, T_U is open in Y. If $T_{\mathcal{U}}$ is not locally finite there are $y \in Y$, a sequence $y_n \to y$ $(n \to \infty)$, and distinct U_n's in $\mathcal{U}$ such that $y_n \in T_{U_n}$. Let $x_n \in \text{supp}(y_n) \cap U_n$. Since $\{y_n : n \in \mathbb{N}\}$ is bounded, by proposition 1.2.8, $\text{supp}\{y_n : n \in \mathbb{N}\}$ is bounded. Hence $\{x_n : n \in \mathbb{N}\}$ is bounded. Since X is metric we then obtain that $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact. Since $\mathcal{U}$ is locally finite, $\overline{\{x_n : n \in \mathbb{N}\}}$ intersects only finitely many elements of $\mathcal{U}$. Contradiction. This proves the lemma. □

Proposition 1.2.8, corollary 1.2.9, lemmas 1.2.10 and 1.2.11 are the first results in this section which are only formulated for linear functions from $C(X)$ to $C(Y)$ and which are not formulated for linear functions from $C^*(X)$ to $C^*(Y)$. In the following example we show that proposition 1.2.8, lemmas 1.2.10 and 1.2.11 are not true for linear functions between function spaces $C^*(X)$.

1.2.12 EXAMPLE: Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence, say $x_n \to x$ $(n \to \infty)$. Let $Y = \{x_n : n \in \mathbb{N}\} \cup \{x\}$, and let $X = Y \oplus \mathbb{N}$. Define $\phi: C^*(X) \to C^*(Y)$ by

$$\phi(f)(z) = \begin{cases} f(x_n) + \frac{1}{n} f(n) & \text{if } z = x_n \text{ for some } n \in \mathbb{N}, \\ f(x) & \text{if } z = x. \end{cases}$$

We first show that ϕ is well-defined. It suffices to show that $\phi(f)$ is continuous at x. Let $\varepsilon > 0$. Since f is bounded there is $c \in \mathbb{R}$ such that $f(X) \subset (-c, c)$. Find $m \in \mathbb{N}$ such that for $n \geq m$, $|f(x_n) - f(x)| < \varepsilon/2$ and $1/m < \varepsilon/(2c)$. Then for $n \geq m$ we have

$$|\phi(f)(x_n) - \phi(f)(x)| = |f(x_n) + \tfrac{1}{n} f(n) - f(x)|$$

$$\leq |f(x_n) - f(x)| + \tfrac{1}{n} |f(n)|$$

$$< \frac{\varepsilon}{2} + \frac{1}{n}c < \varepsilon.$$

So $\phi(f)$ is continuous at x. Note that since Y is compact, $\phi(f) \in C^*(Y)$. Obviously ϕ is linear. As in example 1.2.2 we can show that for each $n \in \mathbb{N}$, $\text{supp}(x_n) = \{n, x_n\}$ and $\text{supp}(x) = \{x\}$. Furthermore for any two functions $f, g \in C^*(X)$ which coincide on $\text{supp}(z)$ for some $z \in Y$, we have $\phi(f)(z) = \phi(g)(z)$. Hence ϕ is a linear function of bounded type.

Since Y is compact, Y is bounded. However $\text{supp}\, Y = X$ is not bounded. This implies that proposition 1.2.8 is not true when dealing with bounded functions. Note also that if $\mathcal{U} = \{Y\} \cup \{\{n\} : n \in \mathbb{N}\}$, then $\mathcal{U}$ is a locally finite open cover of X. However $T_{\mathcal{U}}$ is not locally finite, hence lemma 1.2.11 is also not valid when dealing with bounded functions. Similarly lemma 1.2.10 does not hold. The question remains whether corollary 1.2.9 holds. The above example does not give a counterexample since ϕ is not injective.

We will now search for linear functions of bounded type.

1.2.13 LEMMA: *Let X and Y be spaces. Suppose $\phi: C_0(X) \to C_p(Y)$ (resp. $\phi: C_0^*(X) \to C_p^*(Y)$) is a continuous linear function. Then for every $y \in Y$, $\text{supp}(y)$ is compact.*

PROOF: Since ϕ is continuous at 0, there are a compact $B \subset X$ and $\varepsilon > 0$ such that $\phi(<0, B, \varepsilon>) \subset <0, \{y\}, 1>$. Suppose there is $x \in \text{supp}(y) \setminus B$. In this situation $X \setminus B$ is a neighborhood of x, so there is $f \in C(X)$ (resp. $f \in C^*(X)$) satisfying $f(B) \subset \{0\}$ and $\phi(f)(y) \neq 0$. By linearity of ϕ we may assume $\phi(f)(y) > 1$. Obviously $f \in <0, B, \varepsilon>$, hence $\phi(f) \in <0, \{y\}, 1>$. This implies $\phi(f)(y) < 1$. Contradiction. We conclude that $\text{supp}(y) \subset B$. By lemma 1.2.1 (b) $\text{supp}(y)$ is closed and hence compact. □

1.2.14 PROPOSITION ([1]): *Let X and Y be spaces. Suppose $\phi: C_0(X) \to C_p(Y)$ (resp. $\phi: C_0^*(X) \to C_p^*(Y)$) is a continuous linear function. Then ϕ is of bounded type.*

PROOF: By lemma 1.2.13, $\text{supp}(y)$ is bounded for every $y \in Y$, so it remains to prove that ϕ is effective. If ϕ is not effective, then by lemma 1.2.3 there are $y \in Y$, a neighborhood U of $\text{supp}(y)$ and $f \in C(X)$ (resp. $f \in C^*(X)$) with $f(U) = \{0\}$ and $\phi(f)(y) \neq 0$. Let $\delta = |\phi(f)(y)|$. Since ϕ is continuous, there are a compact subset A of X and $\varepsilon > 0$ such that $\phi(<f, A, \varepsilon>) \subset <\phi(f), \{y\}, \delta>$. Then for every $g \in C(X)$ (resp. $g \in C^*(X)$) which coincides with f on A, $\phi(g)(y) \neq 0$.

Let $B = A \setminus U$. If $B = \emptyset$, $A \subset U$. Since $f(U) = \{0\}$, 0 coincides with f on A. This gives

$\phi(0)(y) \neq 0$. Contradiction, so B is a non-empty compactum and $B \cap \operatorname{supp}(y) = \emptyset$. Then by lemma 1.2.1 (a), there are open sets $U_1, \ldots, U_n, V_1, \ldots, V_n$ such that

(1) $U_i \subset \overline{U}_i \subset V_i$ for every $i \leq n$

(2) $B \subset \bigcup_{i=1}^{n} U_i$, and

(3) if $g \in C(X)$ (resp. $g \in C^*(X)$) with $g(X \setminus V_i) = \{0\}$ for some $i \leq n$, then $\phi(g)(y) = 0$.

Since B is compact, there are $\alpha_1, \ldots, \alpha_n \in C^*(X)$ such that for every $i \leq n$, $\alpha_i(\overline{U}_i \cap B) = 1$ and $\alpha_i(X \setminus V_i) \subset \{0\}$ [24, Th. 3.1.7]. Let $\alpha = \max(\Sigma_{i=1}^{n} \alpha_i, 1)$ and $h_i = \alpha_i/\alpha$. Then we have

(4) $h_i(X \setminus V_i) = \{0\}$, and

(5) $\sum_{i=1}^{n} h_i(x) = 1$ for every $x \in B$

Let $h_i^* = h_i \cdot f$ and $h^* = \Sigma_{i=1}^{n} h_i^*$. By (4) we have $h_i^*(X \setminus V_i) = \{0\}$ so that by (3), $\phi(h_i^*)(y) = 0$. This means $\phi(h^*)(y) = 0$.

By (5), for every $x \in B$ we have $h^*(x) = f(x)$. Furthermore for every $x \in U$ we have $h^*(x) = 0 = f(x)$, so h^* and f coincide on A. But then $\phi(h^*)(y) \neq 0$. Contradiction. We conclude that ϕ is effective. □

1.2.15 COROLLARY ([1]): *Let X and Y be spaces. Suppose $\phi: C_p(X) \to C_p(Y)$ or $\phi: C_0(X) \to C_0(Y)$ is a continuous linear function. Then*

(a) ϕ is of bounded type, and

(b) if A is bounded in Y, then $\operatorname{supp} A$ is bounded in X.

If moreover in X every closed and bounded subset is compact, then $\overline{\operatorname{supp} A}$ is compact.

PROOF: Since $C_p(X) \leq C_0(X)$, any linear mapping $\phi: C_p(X) \to C_p(Y)$ or $\phi: C_0(X) \to C_0(Y)$ is also continuous considered as a function from $C_0(X)$ to $C_p(Y)$. Now apply propositions 1.2.8 and 1.2.14. □

By example 1.2.12 we have for bounded functions only the following corollary the proof of which is similar to the one of corollary 1.2.15 (a).

1.2.16 COROLLARY: *Let X and Y be spaces. Suppose $\phi: C_p^*(X) \to C_p^*(Y)$ or $\phi: C_0^*(X) \to C_0^*(Y)$ is a continuous linear function. Then ϕ is of bounded type.* □

1.2.17 REMARK: The question remains whether a result such as corollary 1.2.15 holds for continuous linear functions between the function spaces $C_b(X)$. In example 1.2.4 (2) we found spaces X and Y and a linear function $\phi: C(X) \to C(Y)$ which is not effective. From corollary 1.2.15 we have that ϕ considered as a function from $C_p(X)$ to $C_p(Y)$ (resp. from $C_0(X)$ to $C_0(Y)$) is not continuous (this can also be verified directly). Unfortunately ϕ considered as a function from $C_b(X)$ to $C_b(Y)$ is also not continuous.

Corollary 1.2.15 will be one of the main tools in this monograph. Another important tool will be corollary 1.2.21. Before we can prove this corollary we need some other lemmas.

1.2.18 LEMMA ([1]): *Let X and Y be spaces, and suppose $\phi: C_u^*(X) \to C_p(Y)$ is a continuous linear function. Then ϕ considered as a function from $C_u^*(X)$ to $C_0(Y)$ is also continuous.*

PROOF: By linearity of ϕ and since $C_0(Y)$ is a topological vector space it suffices to prove continuity at 0, i.e., we have to prove for a compact $A \subset Y$ and $\varepsilon > 0$ that $\phi^{-1}(<0, A, \varepsilon>)$ is a neighborhood of 0 in $C_u^*(X)$. We will show that for

$$V = \{g \in C^*(X) : |\phi(g)(x)| \leq \varepsilon/2 \text{ for every } x \in A\}$$

we have $0 \in \operatorname{Int} V$.

Since $\phi: C_u^*(X) \to C_p(Y)$ is continuous, $\{g \in C^*(X) : |\phi(g)(x)| \leq \varepsilon/2\}$ is closed in $C_u^*(X)$ for every $x \in X$. This means that

$$V = \bigcap_{x \in A} \{g \in C^*(X) : |\phi(g)(x)| \leq \varepsilon/2\}$$

is closed in $C_u^*(X)$.

CLAIM: $C^*(X) = \bigcup_{n \in \mathbb{N}} n \cdot V$.

Let $h \in C^*(X)$. Since A is compact, there is $n_0 \in \mathbb{N}$ such that $\phi(h)(A) \subset [-n_0, n_0]$. Find $n_1 \in \mathbb{N}$ such that $n_1 \geq 2n_0/\varepsilon$. Then for every $x \in A$ we have

$$|\phi(\frac{1}{n_1} h)(x)| = \frac{1}{n_1} |\phi(h(x))| \leq \frac{n_0}{n_1} \leq \frac{\varepsilon}{2}.$$

This means $h \in n_1 \cdot V$ and hence the claim is proved.

Since for every $n \in \mathbb{N}$, $n \cdot V$ is closed in $C_u^*(X)$ and $C_u^*(X)$ is a Banach space, there is $n \in \mathbb{N}$ such that $\operatorname{Int}(n \cdot V) \neq \emptyset$. This means $\operatorname{Int} V \neq \emptyset$. Take an arbitrary $g \in \operatorname{Int} V$. Since

$\theta: C_u^*(X) \to C_u^*(X)$ defined by $\theta(h) = -h$ is a homeomorphism such that $\theta(V) = V$, we have $-g \in \operatorname{Int} V$. Since $\psi: C_u^*(X) \to C_u^*(X)$ defined by $\psi(h) = (g+h)/2$ is a homeomorphism such that $\psi(\operatorname{Int} V) \subset \operatorname{Int} V$ we have $\psi(-g) = 0 \in \operatorname{Int} V$. □

1.2.19 PROPOSITION ([1]): *Let X and Y be spaces such that in X every closed and bounded subset is compact. Let $\phi: C_0(X) \to C_p(Y)$ be a continuous linear function. Then ϕ considered as a function from $C_0(X)$ to $C_0(Y)$ is also continuous.*

PROOF: Let ϕ^* be the restriction of ϕ to the set of bounded functions. Since $C_0^*(X) \leq C_u^*(X)$, ϕ^* considered as a function from $C_u^*(X)$ to $C_p(Y)$ is continuous. Then by lemma 1.2.18, ϕ^* considered as a map from $C_u^*(X)$ to $C_0(Y)$ is continuous. To prove continuity of ϕ considered as a map from $C_0(X)$ to $C_0(Y)$ it is by linearity of ϕ enough to prove continuity at 0. To this end let A be a compact subset of Y and let $\varepsilon > 0$. By propositions 1.2.8 and 1.2.14 and the assumption on the space X, $B = \overline{\operatorname{supp} A}$ is a compact subset of X. By the above there is $\delta > 0$ such that for every $f \in C(X)$ with $|f(x)| < \delta$ for every $x \in X$ we have $\phi(f) \in \langle 0, A, \varepsilon \rangle$. We claim that $\phi(\langle 0, B, \delta/2 \rangle) \subset \langle 0, A, \varepsilon \rangle$. To this end let $g \in C(X)$ with $|g(x)| < \delta/2$ for every $x \in B$.

Define $g_1: X \to \mathbb{R}$ by

$$g_1(x) = \begin{cases} g(x) & \text{if } |g(x)| < \frac{\delta}{2} \\ \frac{\delta}{2} & \text{if } g(x) \geq \frac{\delta}{2} \\ -\frac{\delta}{2} & \text{if } g(x) \leq -\frac{\delta}{2}. \end{cases}$$

Then $g_1 \in C(X)$ and g_1 coincides on a neighborhood of B with g. By proposition 1.2.14, ϕ is effective, and hence $\phi(g) = \phi(g_1)$ on A. Furthermore for every $x \in X$, $|g_1(x)| < \delta$, so $\phi(g_1) \in \langle 0, A, \varepsilon \rangle$. This means $\phi(g) \in \langle 0, A, \varepsilon \rangle$. This proves that ϕ is continuous at 0. □

As with proposition 1.2.8 we have that proposition 1.2.19 does not hold for linear functions between function spaces $C^*(X)$. We have the following

1.2.20 EXAMPLE: Consider X, Y and $\phi: C^*(X) \to C^*(Y)$ as in example 1.2.12. We claim that ϕ considered as a function from $C_p^*(X)$ to $C_p^*(Y)$ is continuous and considered as a function from $C_0^*(X)$ to $C_0^*(Y)$ is not continuous.

First let $P \subset Y$ be finite, $\varepsilon > 0$ and put $Q = \operatorname{supp} P$. Then

$$\phi(<0, Q, \tfrac{\varepsilon}{2}>) \subset <0, P, \varepsilon>$$

Second consider the open subset $<0, Y, 1>$ of $C_0^*(Y)$. If ϕ considered as a function from $C_0^*(X)$ to $C_0^*(Y)$ is continuous, there is a compact subset A of X and $\varepsilon > 0$ such that

$$\phi(<0, A, \varepsilon>) \subset <0, Y, 1>.$$

Find $n \in X \setminus A$. Let $f = n\chi_n$, where χ_n denotes the characteristic function of the set $\{n\}$. Then $f \in <0, A, \varepsilon>$ and $\phi(f)(x_n) = 1$. But this implies that $\phi(f) \notin <0, Y, 1>$.

From proposition 1.2.19 we have the following important

1.2.21 COROLLARY ([1]): *Let X and Y be spaces in which every closed and bounded subset is compact, and suppose $\phi: C_p(X) \to C_p(Y)$ is a linear homeomorphism. Then ϕ considered as a function from $C_0(X)$ to $C_0(Y)$ is also a linear homeomorphism.* □

The converse implication in corollary 1.2.21 is not true. By Miljutin's theorem (for any two uncountable metrizable compact spaces we have that $C_0(X)$ and $C_0(Y)$ are linearly homeomorphic, [47, Th. 21.5.10]), $C_0(\mathrm{I})$ and $C_0(\mathrm{I}^2)$ are linearly homeomorphic (here I denotes the unit interval). However $C_p(\mathrm{I})$ and $C_p(\mathrm{I}^2)$ are not linearly homeomorphic, since Pestov proved in [44] that whenever $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic then $\dim X = \dim Y$.

REMARK: Note that in a compact or metric space the closed and bounded subsets are exactly the compacta. It is not clear to us how this property is related to other topological properties.

For linear mappings between function spaces $C^*(X)$ we can derive a result in the spirit of proposition 1.2.19. This result (corollary 1.2.23) is a consequence of

1.2.22 THE CLOSED GRAPH THEOREM: *Let E and F be Banach spaces and let $\phi: E \to F$ be a linear function such that the set $\{(x, \phi(x)) : x \in E\}$ is closed in $E \times F$. Then ϕ is continuous.*

For a proof of The Closed Graph Theorem we refer to [26].

1.2.23 COROLLARY: *Let X and Y be spaces and let ϕ: $C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. Then ϕ considered as a function from $C_u^*(X)$ to $C_u^*(Y)$ is also continuous.*

PROOF: Since $C_p^*(X) \leq C_u^*(X)$ and $C_p^*(Y) \leq C_u^*(Y)$, it follows directly from The Closed Graph Theorem. □

In corollary 1.2.23 we can replace the topology of pointwise convergence by other topologies. However as stated above it is the only corollary we need in the sequel of this monograph.

REMARK: The results in this section due to Arhangelskiĭ are not formulated in the most general form as they are in [1]. We adjusted these results and their proofs to the form in which we need them in this monograph. The original proof of lemma 1.2.18 used notions like absorbing, convex, circled and balanced spaces (for definitions see [45]). For us these notions are of no importance. Arhangelskiĭ did not define supports for linear functions between function spaces $C^*(X)$. We do not know whether all other results in this section were already known to Arhangelskiĭ.

§1.3. The dual of $C_p(X)$ and $C_p^*(X)$

For a space X let $L(X)$ be the dual of $C_p(X)$, i.e., the set of all continuous linear functionals on $C_p(X)$. For $x \in X$ we define ξ_x: $C_p(X) \to \mathbb{R}$ ***the evaluation mapping*** at x by $\xi_x(f) = f(x)$.

1.3.1 LEMMA: *For every $x \in X$, $\xi_x \in L(X)$.*

PROOF: It is easily seen that ξ_x is linear. To prove that ξ_x is continuous let $U \subset \mathbb{R}$ be open and let $f \in \xi_x^{-1}(U)$. Then $\xi_x(f) = f(x) \in U$. Find $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \subset U$. We claim that $\langle f, \{x\}, \varepsilon \rangle \subset \xi_x^{-1}(U)$. Indeed for $g \in \langle f, \{x\}, \varepsilon \rangle$, $|g(x) - f(x)| < \varepsilon$, so that $g(x) = \xi_x(g) \in U$. □

By identifying x and ξ_x we regard X as a subset of $L(X)$ (notice that for $x \neq y$,

$\xi_x \neq \xi_y$). As $L(X)$ is a vector space we are interested in a Hamel basis for $L(X)$ (i.e., a maximal independent subset). It turns out that X is a Hamel basis for $L(X)$, i.e.,

(HB1) X is an independent subset of $L(X)$ and

(HB2) for every $F \in L(X)$ there are $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $F = \Sigma_{i=1_n} \lambda_i x_i$.

To verify (HB1) suppose $\Sigma_{i=1}^n \lambda_i x_i = 0$ for $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then for every $f \in C(X)$, $\Sigma_{i=1}^n \lambda_i f(x_i) = 0$. For every $i \leq n$ let f_i be a Urysohn function such that $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for $i \neq j$. So $0 = \Sigma_{k=1}^n \lambda_k f_i(x_i) = \lambda_i$, which proves (HB1). For (HB2) we have to do some more work.

1.3.2 LEMMA ([45, p. 124]): *Let V be a vector space and $\alpha, \alpha_1, \ldots, \alpha_n$ linear functionals on V. Then the following statements are equivalent:*

(1) $\bigcap_{i=1}^n \operatorname{Ker} \alpha_i \subset \operatorname{Ker} \alpha$.

(2) $\alpha \in \operatorname{span} \{\alpha_1, \ldots, \alpha_n\}$.

PROOF: The implication (2)⇒(1), is a triviality. We prove the implication (1)⇒(2) by induction on n. First suppose $n = 1$. If $\alpha_1 \equiv 0$, $\alpha \equiv 0$ so we certainly have $\alpha \in \operatorname{span} \{\alpha_1\}$. So suppose there is $x_0 \in V$ such that $\alpha_1(x_0) \neq 0$. Let $\lambda_1 = \alpha(x_0)/\alpha_1(x_0)$. We claim that $\alpha = \lambda_1 \alpha_1$. To prove this, let $x \in V$. If $\alpha_1(x) = 0$ we are done, so suppose $\alpha_1(x) \neq 0$. Then

$$\alpha_1\left(x - \frac{\alpha_1(x)}{\alpha_1(x_0)} x_0\right) = \alpha_1(x) - \frac{\alpha_1(x)}{\alpha_1(x_0)} \alpha_1(x_0) = 0,$$

so

$$x - \frac{\alpha_1(x)}{\alpha_1(x_0)} x_0 \in \operatorname{Ker} \alpha_1 \subset \operatorname{Ker} \alpha.$$

This gives

$$0 = \alpha\left(x - \frac{\alpha_1(x)}{\alpha_1(x_0)} x_0\right) = \alpha(x) - \frac{\alpha_1(x)}{\alpha_1(x_0)} \alpha(x_0) = \alpha(x) - \lambda \alpha_1(x).$$

So $\alpha(x) = \lambda_1 \alpha_1(x)$. This finishes the case $n = 1$.

Suppose we proved the implication for every $n < m$ with $m > 1$.

Case 1: there is $j \leq m$ with $\bigcap_{i \neq j} \operatorname{Ker} \alpha_i \subset \operatorname{Ker} \alpha$.

Then by the inductive hypothesis, $\alpha \in \operatorname{span} \{\alpha_1, \ldots, \alpha_m\}$.

Case 2: for every $j \leq m$, $\bigcap_{i \neq j} \operatorname{Ker} \alpha_i \not\subset \operatorname{Ker} \alpha$.

Then for every $j \leq m$ there is $x_j \in V$ such that $\alpha_i(x_j)=0$ for $i \neq j$ and $\alpha_j(x_j) \neq 0$. Let $\lambda_j = \alpha(x_j)/\alpha_j(x_j)$. We claim that $\alpha = \Sigma_{i=1}^{n} \lambda_i \alpha_i$. To prove this let $x \in V$. For $j \leq m$ we have

$$\alpha_j(x - \sum_{i=1}^{m} \frac{\alpha_i(x)}{\alpha_i(x_i)} x_i) = \alpha_j(x) - \frac{\alpha_j(x)}{\alpha_j(x_j)} \alpha_j(x_j) = 0.$$

This gives

$$x - \sum_{i=1}^{m} \frac{\alpha_i(x)}{\alpha_i(x_i)} x_i \in \bigcap_{i=1}^{m} \operatorname{Ker} \alpha_i \subset \operatorname{Ker} \alpha.$$

Hence

$$\alpha(x - \sum_{i=1}^{m} \frac{\alpha_i(x)}{\alpha_i(x_i)} x_i) = \alpha(x) - \sum_{i=1}^{m} \lambda_i \alpha_i(x) = 0.$$

This finishes the proof of this lemma. □

1.3.3 THEOREM: *X is a Hamel basis for $L(X)$.*

PROOF: As mentioned above it is enough to prove condition (HB2). So let $F: C_p(X) \to \mathbb{R}$ be a continuous linear functional. There is a finite subset P of X and $\delta > 0$ such that $F(<0, P, \delta>) \subset (-1, 1)$. Suppose $P = \{x_1, \ldots, x_n\}$. We claim that $\bigcap_{i=1}^{n} \operatorname{Ker} \xi_{x_i} \subset \operatorname{Ker} F$. Indeed let $f \in \bigcap_{i=1}^{n} \operatorname{Ker} \xi_{x_i}$. Then for every $i \leq n$, $f(x_i)=0$. Let $\varepsilon > 0$. Clearly $(1/\varepsilon) \cdot f \in <0, P, \delta>$, so that $F((1/\varepsilon) \cdot f) \subset (-1, 1)$ or, equivalently $F(f) \subset (-\varepsilon, \varepsilon)$. Since ε was arbitrary we have $F(f)=0$ which implies $f \in \operatorname{Ker} F$. Now by lemma 1.3.2, $F \in \operatorname{span} \{\xi_{x_1}, \ldots, \xi_{x_n}\}$. □

We can define a topology on $L(X)$ as follows. For $f \in C(X)$ let $L(f): L(X) \to \mathbb{R}$ be defined by $L(f)(F) = F(f)$. The topology on $L(X)$ is the weakest topology which makes all $L(f)$ ($f \in C_p(X)$) continuous, i.e., the topology which has as a subbase the family,

$$\{L(f)^{-1}(U) : f \in C(X) \text{ and U open in } \mathbb{R}\}.$$

With this topology, $L(X)$ is called the topological dual of $C_p(X)$. Clearly $L(X)$ is then a locally convex topological vector space.

1.3.4 LEMMA: *Let $f \in C_p(X)$. Then $L(f): L(X) \to \mathbb{R}$ is the unique continuous linear functional that extends f.*

PROOF: That $L(f)$ is a continuous linear functional is obvious. For $x \in X$ we have

$L(f)(x)=x(f)=f(x)$, so that $L(f)$ extends f. Since X is a Hamel basis for $L(X)$ it follows that $L(f)$ is unique. □

In theorem 1.3.3 we derived that X is algebraically a special subset of $L(X)$. Topologically we have

1.3.5 PROPOSITION: *X is homeomorphic to X as subspace of $L(X)$, and X as a subspace of $L(X)$ is closed in $L(X)$.*

PROOF: First of all we show that X is homeomorphic to X as subspace of $L(X)$. Let U be open in X and $x \in U$. Let $f \in C(X)$ be a Urysohn function such that $f(x)=1$ and $f(X \setminus U)=\{0\}$. By lemma 1.3.4, f extends to a continuous linear functional $L(f): L(X) \to \mathbb{R}$. Let $V=L(f)^{-1}(0,\infty)$. Then V is open in $L(X)$ and $x \in V \cap X \subset U$. Now let V be open in $L(X)$ and let $x \in V \cap X$. There are $f_1, \ldots, f_n \in C_p(X)$ and $U_1, \ldots, U_n$ open in $\mathbb{R}$ with $x \in \bigcap_{i=1}^n L(f_i)^{-1}(U_i) \subset V$. Then $x \in \bigcap_{i=1}^n f_i^{-1}(U_i) \subset V \cap X$. We conclude that X is a subspace of $L(X)$.

Second we show that X as subspace of $L(X)$ is closed in $L(X)$. To this end let $F \in L(X) \setminus X$. By theorem 1.3.3 there are $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $F=\Sigma_{i=1}^n \lambda_i x_i$ with $x_i \neq x_j$ for $i \neq j$.

Case 1: $n \geq 2$ and $\lambda_i \neq 0$ for all $i \leq n$.

For each $i \leq n$ find V_i open in X and U_i open in $\mathbb{R}$ such that $V_i \cap V_j=\emptyset$ $(i \neq j)$, $x_i \in V_i$, $\lambda_i \in U_i$ and $0 \notin U_i$. For each $i \leq n$ there is a Urysohn function $f_i: X \to \mathbb{R}$ such that $f_i(x_i)=1$ and $f_i(X \setminus V_i)=0$. By lemma 1.3.4, f_i has a continuous linear extension $L(f_i): L(X) \to \mathbb{R}$. We claim that $F \in \bigcap_{i=1}^n L(f_i)^{-1}(U_i) \subset L(X) \setminus X$. Indeed for each $i \leq n$,

$$L(f_i)(F)=\sum_{j=1}^n \lambda_j L(f_i)(x_j)=\sum_{j=1}^n \lambda_j f_i(x_j)=\lambda_i \in U_i$$

and for $x \in X$ there is $i \leq n$ with $x \notin V_i$ (since $n \geq 2$) so that $L(f_i)(x)=f_i(x) \equiv 0 \notin U_i$. This implies $X \cap \bigcap_{i=1}^n L(f_i)^{-1}(U_i)=\emptyset$, which proves case 1.

Case 2: $F=\lambda_1 x_1$.

Since $F \notin X$, $\lambda_1 \neq 1$. Hence there is U open in $\mathbb{R}$ with $\lambda_1 \in U$ and $1 \notin U$. Let $f \equiv 1$ and $L(f): L(X) \to \mathbb{R}$ be its continuous linear extension. We claim that $F \in L(f)^{-1}(U) \subset L(X) \setminus X$. Indeed

$$L(f)(F)=\lambda_1 L(f)(x_1)=\lambda_1 f(x_1)=\lambda_1 \in U$$

and for $x \in X$ we have $L(f)(x) = f(x) = 1 \notin U$. This proves case 2 and the proposition. □

The following proposition will be of importance in the last section of this chapter.

1.3.6 PROPOSITION: *Let X and Y be spaces. Then $C_p(X) \sim C_p(Y)$ if and only if $L(X) \sim L(Y)$.*

PROOF: First suppose $\phi: C_p(X) \to C_p(Y)$ is a linear homeomorphism. Define $\psi: L(X) \to L(Y)$ by $\psi(F) = F \circ \phi^{-1}$. Then ψ is obviously a well-defined linear function. To see that ψ is continuous notice that for $f \in C_p(Y)$ and $U \subset \mathbb{R}$ open we have

$$\psi^{-1}(L(f)^{-1}(U)) = (L(f) \circ \psi)^{-1}(U) = (L(\phi^{-1}(f))^{-1}(U)$$

is open in $L(X)$.

Define $\theta: L(Y) \to L(X)$ by $\theta(G) = G \circ \phi$. In the same way we can prove that θ is a well-defined linear mapping. As is easily seen $\theta = \psi^{-1}$, so that ψ is a linear homeomorphism.

Second, suppose $\psi: L(X) \to L(Y)$ is a linear homeomorphism. Define $\phi: C_p(X) \to C_p(Y)$ by $\phi(f) = (L(f) \circ \psi^{-1}) | Y$. Then ϕ is obviously a well-defined linear function. In order to prove that ϕ is continuous at 0 let $P \subset Y$ be finite and $\varepsilon > 0$. For every $y \in Y$ there are $x_1^y, \ldots, x_{n_y}^y \in X$ and $\lambda_1^y, \ldots, \lambda_{n_y}^y \in \mathbb{R} \setminus \{0\}$ such that

$$\psi^{-1}(y) = \sum_{i=1}^{n_y} \lambda_i^y x_i^y.$$

Let $N = \max\{\Sigma_{i=1}^{n_y} |\lambda_i^y| : y \in P\}$, let $\delta = \varepsilon/N$ and let $Q = \{x_i^y : y \in P \text{ and } i \leq n_y\}$. We claim that $\phi(<0, Q, \delta>) \subset <0, P, \varepsilon>$. Indeed, if $f \in <0, Q, \delta>$ we have for $y \in P$

$$|\phi(f)(y)| = |(L(f) \circ \psi^{-1})(y)| = |L(f)(\sum_{i=1}^{n_y} \lambda_i^y x_i^y)| = |\sum_{i=1}^{n_y} \lambda_i^y f(x_i^y)|$$
$$\leq \sum_{i=1}^{n_y} |\lambda_i^y| \, |f(x_i^y)| < \delta \sum_{i=1}^{n_y} |\lambda_i^y| \leq \varepsilon$$

By linearity of ϕ we conclude that ϕ is continuous.

Define $\theta: C_p(Y) \to C_p(X)$ by $\theta(g) = (L(g) \circ \psi) | X$. In the same way we can prove that θ is a well-defined linear mapping and as is easily seen, $\theta = \phi^{-1}$ so that ϕ is a linear homeomorphism. □

We define $L^*(X)$ similar to $L(X)$ using $C_p^*(X)$ instead of $C_p(X)$. All observations made above for $L(X)$ are also valid for $L^*(X)$.

REMARK: The results in this section are often used in the literature on function spaces with the topology of pointwise convergence (for example [41] and [44]). We were not able to find an explicit reference for their proofs, so we provided them ourselves. For more information on dual spaces of topological vector spaces we refer to [45].

§1.4. Supports and the topology of pointwise convergence

When dealing with continuous linear functions between function spaces endowed with the topology of pointwise convergence, it is possible to give a precise description of supports (cf. lemma 1.4.1).

Let X and Y be spaces, let $\phi: C_p(X) \to C_p(Y)$ (resp. $\phi: C_p^*(X) \to C_p^*(Y)$) be a continuous linear function and let $y \in Y$ be fixed. Notice that the function $\psi_y: C_p(X) \to \mathbb{R}$ (resp. $\psi_y: C_p^*(X) \to \mathbb{R}$) defined by $\psi_y = \xi_y \circ \phi$ is continuous and linear. So $\psi_y \in L(X)$ (resp. $\psi_y \in L^*(X)$), the dual of $C_p(X)$ (resp. $C_p^*(X)$). For every $f \in C(Y)$ (resp. $f \in C^*(X)$) we have $\psi_y(f) = \phi(f)(y)$. By theorem 1.3.3 there are for $\psi_y \neq 0$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$ such that $\psi_y = \Sigma_{i=1}^n \lambda_i x_i$ (notice that whenever ϕ is a bijection, $\psi_y \neq 0$ for every $y \in Y$). This means that for every $f \in C(X)$ (resp. $f \in C^*(X)$), $\phi(f)(y) = \Sigma_{i=1}^n \lambda_i f(x_i)$. Then

1.4.1 LEMMA: $\operatorname{supp}(y) = \{x_1, \ldots, x_n\}$.

PROOF: Let $x \in \operatorname{supp}(y)$ and suppose that $x \notin \{x_1, \ldots, x_n\}$. Since $X \setminus \{x_1, \ldots, x_n\}$ is open, there is $f \in C(X)$ (resp. $f \in C^*(X)$) such that $f(x_i) = 0$ for every $i \leq n$ and $\phi(f)(y) \neq 0$. But $\phi(f)(y) = \Sigma_{i=1}^n \lambda_i f(x_i) = 0$. Contradiction.

Now let $i \leq n$ be fixed and U an open neighborhood of x_i such that $U \cap \{x_j : j \leq n \text{ and } j \neq i\} = \emptyset$. Let $f \in C^*(X)$ be a Urysohn function with $f(X \setminus U) = 0$ and $f(x_i) = 1$. Then $\phi(f)(y) = \Sigma_{i=1}^n \lambda_i f(x_i) = \lambda_i \neq 0$. □

From lemma 1.4.1 we have the following corollary of which part (b) simplifies the notion of effectiveness in the case of the topology of pointwise convergence

1.4.2 COROLLARY: *Let X and Y be spaces, and let $\phi: C_p(X) \to C_p(Y)$ (resp. $\phi: C_p^*(X) \to C_p^*(Y)$), be a continuous linear function. Then for $y \in Y$,*

(a) for every $z \in \operatorname{supp}(y)$, there is $\lambda_z \in \mathbb{R}$ such that $\phi(f)(y) = \Sigma_{z \in \operatorname{supp}(y)} \lambda_z f(z)$,

for every $f \in C(X)$ *(resp.* $f \in C^*(X)$*), and*
(b) if $f, g \in C(X)$ *(resp.* $f, g \in C^*(X)$*), coincide on* $\operatorname{supp}(y)$*, then* $\phi(f)(y) = \phi(g)(y)$.

Another useful property of supports with respect to the topology of pointwise convergence is given in the following:

1.4.3 PROPOSITION: *Let* X *and* Y *be spaces and let* $\phi: C_p(X) \to C_p(Y)$ *(resp.* $\phi: C_p^*(X) \to C_p^*(Y)$*) be a linear homeomorphism. Then for every* $x \in X$ *we have* $x \in \operatorname{supp}\operatorname{supp}(x)$ *(in other words, for every* $x \in X$ *there is* $y \in \operatorname{supp}(x)$ *such that* $x \in \operatorname{supp}(y)$*). In particular* $\operatorname{supp} Y = X$.

PROOF: Let $x \in X$ and suppose $x \notin \operatorname{supp}\operatorname{supp}(x)$. Since $\operatorname{supp}\operatorname{supp}(x)$ is finite (lemma 1.4.1), there is a Urysohn function $f \in C^*(X)$ such that $f(x) = 1$ and $f(\operatorname{supp}\operatorname{supp}(x)) = 0$. By corollary 1.4.2 (b) it follows that $\phi(f) = 0$ on $\operatorname{supp}(x)$ and again by corollary 1.4.2 (b) it then follows that $f(x) = \phi^{-1}(\phi(f))(x) = 0$, and we arrived at a contradiction. □

1.4.4 PROPOSITION: *Let* X *and* Y *be spaces and let* $\phi: C_p(X) \to C_p(Y)$ *(resp.* $\phi: C_p^*(X) \to CP^*Y$*) be a continuous linear surjection. Then* $\operatorname{supp}: Y \to \mathcal{P}(X) \setminus \{\emptyset\}$ *is LSC.*

PROOF: This follows from corollary 1.2.15 (a) (resp. corollary 1.2.16), lemma 1.2.7 and the remark following lemma 1.2.7. □

In section 1.5 we need the following

1.4.5 LEMMA: *Let* X *and* Y *be normal spaces, and let* $\phi: C_p(X) \to C_p(Y)$ *be a continuous linear surjection. Then for each closed and bounded* $K \subset X$*, the set* $L = \{y \in Y : \operatorname{supp}(y) \subset K\}$ *is closed and bounded in* Y.

PROOF: By proposition 1.4.4, supp is LSC hence L is closed. If L is not bounded, L contains a closed discrete subset $\{y_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $t_n = n \cdot \Sigma_{z \in \operatorname{supp}(y_n)} |\lambda_z|$. Then $t_n > 0$. Let $g \in C(Y)$ be such that $g(y_n) = t_n$. Since ϕ is a surjection, there is $f \in C(X)$ such that $\phi(f) = g$. Since K is bounded, there is $c \in \mathbb{R}$ such that $f(K) \subset [-c, c]$.

Let $n \in \mathbb{N}$ be such that $n > c$. Then

$$\begin{aligned} |\phi(f)(y_n)| &= |\Sigma_{z\in \mathrm{supp}(y_n)}\lambda_z f(z)| \\ &\leq \Sigma_{z\in \mathrm{supp}(y_n)} |\lambda_z| \cdot |f(z)| \\ &\leq c\cdot\Sigma_{z\in \mathrm{supp}(y_n)} |\lambda_z| \\ &< t_n. \end{aligned}$$

Contradiction. This proves the lemma. □

If we consider function spaces $C_p^*(X)$ we have the weaker

1.4.6 LEMMA: *Let X and Y be metric spaces and let $\phi: C_p^*(X) \to C_p^*(Y)$ be a continuous linear surjection. Then for each compact $K \subset X$, the set $L = \{y \in Y : \mathrm{supp}(y) \subset K\}$ is compact.*

PROOF: By proposition 1.4.4, supp is LSC, hence L is closed in Y. For $f \in C^*(K)$, let $\tilde{f} \in C^*(X)$ be an extension of f. Define

$$\theta: C_p^*(K) \to C_p^*(L) \text{ by } \theta(f) = \phi(\tilde{f})|L.$$

If $g \in C^*(X)$ is another extension of f, then $\tilde{f}$ and g coincides on $\mathrm{supp}L$, hence by corollary 1.4.2 (b) $\phi(\tilde{f}) = \phi(g)$ on L. This implies that θ is well-defined. It follows that θ is a continuous linear function. By corollary 1.2.23 we then have that θ considered as a function from $C_u^*(K)$ to $C_u^*(L)$ is also continuous. We claim that θ is surjective. Let $g \in C^*(L)$ and let $\tilde{g} \in C^*(Y)$ be an extension of g. Since ϕ is surjective, there is $h \in C^*(X)$ with $\phi(h) = \tilde{g}$. Let $f = h|K$. Since h extends f, $\theta(f) = g$, so θ is a surjection. By [47, Prop. 7.6.2] we have for a space Z that $C_u^*(Z)$ is separable if and only if Z is compact and metrizable. This implies that $C_u^*(K)$ is separable and hence $C_u^*(L)$ is separable. So L is compact. □

The proofs of lemma 1.4.5 and lemma 1.4.6 are different and not reversible.

REMARK: Jan Pelant provided us with the description of supports when dealing with the topology of pointwise convergence. We were informed that Arhangelskiĭ knew of this description of supports.

§1.5. First applications

Let X and Y be spaces. We define X and Y to be l_p, l_0 or l_b-***equivalent*** whenever

$C_p(X)$ and $C_p(Y)$, $C_0(X)$ and $C_0(Y)$ or $C_b(X)$ and $C_b(Y)$ are linearly homeomorphic. We say that a topological property $\mathcal{P}$ ***is preserved by*** ι_p, ι_0 or ι_b***-equivalence*** (resp. ι_p^*, ι_0^* or ι_b^****-equivalence***) if for ι_p, ι_0 or ι_b-equivalent (resp. ι_p^*, ι_0^* or ι_b^*-equivalent) spaces X and Y we have X has property $\mathcal{P}$ iff Y has property $\mathcal{P}$. In this section we give some topological properties which are or which are not preserved by ι_p, ι_0 or ι_b-equivalence (resp. ι_p^*, ι_0^* or ι_b^*-equivalence) and we state some questions.

1.5.1 THEOREM [1]: *The following topological properties are preserved by* ι_p*-equivalence:*

(*a*) *pseudocompactness,*

(*b*) *compactness, and*

(*c*) σ*-compactness.*

PROOF: Let X and Y be ι_p-equivalent spaces.

By corollary 1.2.9 and 1.2.15 (a) we have that pseudocompactness is preserved by ι_p-equivalence.

For (b) and (c) we use that by proposition 1.3.6 $L(X)$ and $L(Y)$ are linearly homeomorphic. For every $n \in \mathbb{N}$ define $h_n: X^n \times [-n, n]^n \to L(X)$ by

$$h_n(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i x_i.$$

By proposition 1.3.5, X is homeomorphic to X as subspace of the topological vector space $L(X)$, hence h_n is continuous. Furthermore $L(X) = \bigcup_{i=1}^{\infty} h_n(X_n \times [-n, n]^n)$. Suppose X is σ-compact. Then we have that $L(X)$ is σ-compact, and hence that $L(Y)$ is σ-compact. By proposition 1.3.5, Y is closed in $L(Y)$ so Y is σ-compact. This finishes the proof of (c). When X is compact we again have that Y is σ-compact and hence Lindelöf. Furthermore by (a) we have that Y is pseudocompact. Since each Lindelöf space is normal, and each normal pseudocompact space is countably compact, we have that Y is a Lindelöf countably compact space, hence Y is compact. □

This theorem and the proof of (b) and (c) are due to Arhangelskiĭ. It follows that for normal spaces countable compactness is preserved by ι_p-equivalence. Whether this is true for all spaces is still an open question. By the observations in section 1.2 it was possible to give an easier proof of (a), than the original one. For (b) and (c) this is not possible unless we assume that in X and Y every closed and bounded subset is compact. For such spaces we will now derive in theorem 1.5.2 a result in the same spirit as the previous one.

For a space X let $\mathcal{K}(X)$ be the family of all compact subsets of X. We regard $\mathcal{K}(X)$ as a poset under inclusion. Then a subset $\mathcal{B}$ is cofinal in $\mathcal{K}(X)$ whenever for each $K \in \mathcal{K}(X)$ there is $B \in \mathcal{B}$ with $K \subset B$. The cofinality of $\mathcal{K}(X)$ is the cardinal defined by

$$\operatorname{cof}\mathcal{K}(X) = \min\{\,|\mathcal{B}| : \mathcal{B} \text{ is cofinal in } \mathcal{K}(X)\}$$

(cf. [21]). A space X is said to be ***hemicompact*** whenever $\operatorname{cof}\mathcal{K}(X) \leq \omega$.

1.5.2 THEOREM: *The cofinality of the family of compact subsets of a space is preserved by l_p-equivalence in the class of spaces in which every closed and bounded subset is compact.*

PROOF: Let X and Y be l_p-equivalent spaces in which every closed and bounded subset is compact and let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism. Without loss of generality we assume $\operatorname{cof}\mathcal{K}(X) \leq \operatorname{cof}\mathcal{K}(Y)$. Let $\{K_i : i \in I\}$ be cofinal in $\mathcal{K}(X)$ such that $|I| = \operatorname{cof}\mathcal{K}(X)$. By corollary 1.2.15 (b) and the assumption on Y we have that $\overline{\operatorname{supp} K_i}$ is compact for every $i \in I$. It suffices to prove that $\{\overline{\operatorname{supp} K_i} : i \in I\}$ is cofinal in $\mathcal{K}(Y)$. For this let $A \subset Y$ be compact. Again by corollary 1.2.15 (b) and the assumption on X, $\overline{\operatorname{supp} A}$ is compact in X. So there is $i \in I$ with $\overline{\operatorname{supp} A} \subset K_i$. Then by proposition 1.4.3, $A \subset \operatorname{supp}\operatorname{supp} A \subset \overline{\operatorname{supp} K_i}$. □

It remains open whether in general the cofinality of the family of compact subsets is preserved by l_p-equivalence. The results in section 1.4 allow us to obtain stronger results for an even more restricted class of spaces.

1.5.3 THEOREM: *Let X and Y be normal spaces and let $\phi: C_p(X) \to C_p(Y)$ be a continuous linear surjection.*

(a) If X is pseudocompact, then Y is pseudocompact.

If moreover in Y every closed and bounded subset is compact, then

(b) if X is compact, then Y is compact,

(c) if X is σ-compact, then Y is σ-compact, and

(d) $\operatorname{cof}\mathcal{K}(Y) \leq \operatorname{cof}\mathcal{K}(X)$.

PROOF: For part (a) we have by lemma 1.4.5 that the set $\{y \in Y : \operatorname{supp}(y) \subset X\} = Y$ is pseudocompact whenever X is pseudocompact. Part (b) follows from part (a) and the assumption on Y.

For (c) let $X = \bigcup_{n=1}^{\infty} X_n$ with for each $n \in \mathbb{N}$, $X_n \subset X_{n+1}$ and X_n compact. Let $Y_n = \{y \in Y : \operatorname{supp}(y) \subset X_n\}$. By lemma 1.4.5 and the assumption on Y we have that Y_n

is compact. Since for each $y \in Y$, $\operatorname{supp}(y)$ is finite we also have that $Y = \bigcup_{n=1}^{\infty} Y_n$.

For (d) let $\{K_i : i \in I\}$ be cofinal in $\mathcal{K}(X)$. For each $i \in I$ let $L_i = \{y \in Y : \operatorname{supp}(y) \subset K_i\}$. Then L_i is compact. We claim that $\{L_i : i \in I\}$ is cofinal in $\mathcal{K}(Y)$. Let $A \subset Y$ be compact. Then by corollary 1.2.15 (b), $\overline{\operatorname{supp} A}$ is compact. Hence $\overline{\operatorname{supp} A} \subset K_i$ for some $i \in I$. This implies $A \subset L_i$. □

By using lemma 1.4.6 instead of lemma 1.4.5 we obtain for metric spaces the following

1.5.4 THEOREM: *Let X and Y be metric spaces and let $\phi: C_p^*(X) \to C_p^*(Y)$ be a continuous linear surjection. Then*

(a) if X is compact, then Y is compact, and

(b) if X is σ-compact, then Y is σ-compact. □

1.5.5 COROLLARY: *Compactness and σ-compactness are preserved by ι_p^*-equivalence in the class of metric spaces.* □

The proof of theorem 1.5.3 (d) makes use of corollary 1.2.15 (b). Since we do not have such a result for continuous linear functions between function spaces $C_p^*(X)$ we cannot copy the proof of theorem 1.5.3 (d) to this case.

Now that we have the above theorems for ι_p and ι_p^*-equivalence, we become interested whether the same result hold for ι_0 and ι_0^*-equivalence (resp. ι_b and ι_b^*-equivalence). First we deal with ι_0 and ι_0^*-equivalence.

1.5.6 LEMMA: *Let X and Y be spaces and let $\phi: C_0(X) \to C_0(Y)$ (resp. $\phi: C_0^*(X) \to C_0^*(Y)$) a continuous linear function. Then for every compact $B \subset Y$, $\overline{\operatorname{supp} B}$ is compact.*

PROOF: There are non-empty compacta $C_1, \ldots, C_n$ in X and open $U_1, \ldots, U_n$ in Y such that

$$0 \in \bigcap_{i=1}^{n} A(C_i, U_i) \subset \phi^{-1}(A(B, (-1, 1))).$$

Let $C = \bigcup_{i=1}^{n} C_i$. Then C is compact. We claim that $\operatorname{supp} B \subset C$. To the contrary suppose there are $y \in B$ and $x \in \operatorname{supp}(y) \backslash C$. Since $X \backslash C$ is a neighborhood of x and $x \in \operatorname{supp}(y)$ there is $f \in C(X)$ (resp. $f \in C^*(X)$) such that $f(C) = 0$ and $\phi(f)(y) \neq 0$. By linearity of ϕ we may assume $\phi(f)(y) = 1$. Since $0 \in \bigcap_{i=1}^{n} A(C_i, U_i)$ we have

$0 \in \bigcap_{i=1}^{n} U_i$, so $f \in \bigcap_{i=1}^{n} A(C_i, U_i)$. This gives $\phi(f) \in A(B, (-1, 1))$ which implies that $\phi(f)(y) \in (-1, 1)$. Contradiction. We now have $\operatorname{supp} B \subset C$, so that $\overline{\operatorname{supp} B} \subset C$, which implies that $\overline{\operatorname{supp} B}$ is compact. □

1.5.7 THEOREM: *Pseudocompactness is preserved by ι_0-equivalence. Compactness is preserved by ι_0 and ι_0^*-equivalence.*

PROOF: Let X and Y be ι_0-equivalent spaces By corollary 1.2.9 and 1.2.15 (a) we have that pseudocompactness is preserved by ι_0-equivalence.

Let $\phi: C_0(X) \to C_0(Y)$ (resp. $\phi: C_0^*(X) \to C_0^*(Y)$) be a linear homeomorphism. Suppose X is compact. By lemma 1.2.5 and corollary 1.2.15 (a) (resp. corollary 1.2.16), $\overline{\operatorname{supp} X} = Y$, and hence by lemma 1.5.6, Y is compact. □

The proof that pseudocompactness is preserved by ι_0-equivalence cannot be copied for ι_0^*-equivalence since the proof of corollary 1.2.9 makes use of corollary 1.2.15 (b). From theorem 1.5.7 and corollary 1.2.21, theorem 1.5.1 (a) and (b) follow for the class of spaces in which every closed and bounded subset is compact.

For ι_b and ι_b^*-equivalence we have

1.5.8 THEOREM: *Pseudocompactness is preserved by ι_b and ι_b^*-equivalence.*

PROOF: Let X and Y be ι_b-equivalent (resp. ι_b^*-equivalent) spaces and let $\phi: C_b(X) \to C_b(Y)$ (resp. $\phi: C_b^*(X) \to C_b^*(Y)$) be a linear homeomorphism. Suppose Y is pseudocompact and X is not pseudocompact. Since $<0, Y, 1>$ is open in $C_b(Y)$ (resp. $C_b^*(Y)$) there are $f_1, \ldots, f_n$ in $C(X)$ (resp. $C^*(X)$), bounded $A_1, \ldots, A_n$ in X and $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $0 \in \bigcap_{i=1}^{n} <f_i, A_i, \varepsilon_i>$ and $\phi(\bigcap_{i=1}^{n} <f_i, A_i, \varepsilon_i>) \subset <0, Y, 1>$. Let $A = \bigcup_{i=1}^{n} A_i$. Then A is bounded. Since X is not pseudocompact there is $x \in X \setminus \bar{A}$. Let $f \in C^*(X)$ be a Urysohn function such that $f(A) = 0$ and $f(x) = 1$. Since $f \neq 0$, $\phi(f) \neq 0$. Let $y \in Y$ be such that $\phi(f)(y) \neq 0$. Define $g: X \to \mathbb{R}$ by $g = f/\phi(f)(y)$. Since $g(A) = \{0\}$, $g \in \bigcap_{i=1}^{n} <f_i, A_i, \varepsilon_i>$, so $\phi(g) \in <0, Y, 1>$. However $\phi(g)(y) = 1$. Contradiction. □

Question 1: Is pseudocompactness preserved by ι_0^*-equivalence? Is compactness preserved by ι_b or ι_b^*-equivalence? Are σ-compactness or the cofinality of the family of compact subsets of a space preserved by ι_0, ι_0^*, ι_b or ι_b^*-equivalence?

From now on in this section we only deal with function spaces endowed with the topology of pointwise convergence.

It is well-known that cardinality and density are preserved by ι_p-equivalence [2]. By the above techniques, we can give an alternative proof which is also valid for ι_p^*-equivalence:

1.5.9 THEOREM: *The following cardinal invariants are preserved by ι_p-equivalence (resp. ι_p^*-equivalence),*

(a) cardinality, and

(b) density.

PROOF: Let X and Y be ι_p-equivalent (resp. ι_p^*-equivalent) spaces and let $\phi: C_p(X) \to C_p(Y)$ (resp $\phi: C_p^*(X) \to C_p^*(Y)$) be a linear homeomorphism. For (a), notice that if $|X| = n$, the algebraic dimension of $C_p(X)$ is equal to n, hence we have that $|Y| = n$. So without loss of generality we assume $\aleph_0 \leq |X| \leq |Y|$. By lemma 1.4.1, $|\operatorname{supp} X| \leq |X|$, so by proposition 1.4.3, $|Y| \leq |X|$. We conclude that $|X| = |Y|$.

For (b) notice that if $d(X)$ is finite, then since cardinality is preserved by ι_p-equivalence, $d(X) = |X| = |Y| = d(Y)$. So without loss of generality we assume $\aleph_0 \leq d(X) \leq d(Y)$. Let $D \subset X$ be such that $\overline{D} = X$ and $|D| = d(X)$. Let $E = \operatorname{supp} D$. By lemma 1.4.1, $|E| \leq |D|$. To prove that $d(X) = d(Y)$ it suffices to prove that $\overline{E} = Y$. By proposition 1.4.3, lemma 1.2.6 and corollary 1.2.15 (a) (resp. corollary 1.2.16),

$$Y = \operatorname{supp} X = \operatorname{supp} \overline{D} \subset \overline{\operatorname{supp} D} = \overline{E} \subset Y.$$

We conclude that density is preserved by ι_p-equivalence. □

As a corollary we see that separability is preserved by ι_p and ι_p^*-equivalence so Lindelöfness is preserved by ι_p and ι_p^*-equivalence in the class of metric spaces.

Question 2: Are density or cardinality preserved by ι_0 or ι_0^*-equivalence (resp. ι_b or ι_b^*-equivalence)?

1.5.10 THEOREM: *Local compactness is preserved by ι_p-equivalence in the class of paracompact first countable spaces.*

PROOF: Let X and Y be ι_p-equivalent paracompact first countable spaces. Suppose X is locally compact and Y is not locally compact. Since X is a locally compact paracompact space, there is a locally finite open cover $\{X_s : s \in S\}$ of X such that for each $s \in S$,

$\overline{X}_s$ is compact. Let $y \in Y$ be a point without compact neighborhood and let $\{U_n : n \in \mathbb{N}\}$ be a decreasing neighborhood base at y. Then for each $n \in \mathbb{N}$, $\overline{U}_n$ is not compact.

By lemma 1.2.10 and corollary 1.2.15 (a) there is $k \in \mathbb{N}$ and $\{s_1, \ldots, s_k\} \subset S$ such that $\operatorname{supp} U_k \subset \bigcup_{i=1}^{k} X_{s_i}$. Let $L = \overline{\bigcup_{i=1}^{k} X_{s_i}}$. Then L is compact.

We now have by lemma 1.2.6 and corollary 1.2.15 (a), $\operatorname{supp} \overline{U}_k \subset \overline{\operatorname{supp} U_k} \subset L$ so by proposition 1.4.3, $\overline{U}_k \subset \operatorname{supp} \operatorname{supp} \overline{U}_k \subset \operatorname{supp} L$. Since each countably compact paracompact space is compact [24, Th 5.1.20], we have that $\overline{U}_k$ is not countably compact. Since each paracompact space is normal [24, Th 5.1.18], Y is normal, and since each pseudocompact normal space is countably compact [24, Th. 3.10.21], $\overline{U}_k$ is not pseudocompact. Hence by normality of Y, $\overline{U}_k$ is not bounded in Y. However L is compact so by corollary 1.2.15 (b), $\operatorname{supp} L$ is bounded in Y. Contradiction. □

Theorem 1.5.10 is due to S.P. Gulko and O.G. Okunev [2]. Their proof was by different methods than ours. In section 2.4 we show that the first countability assumption is essential in this result.

Question 3: Is paracompactness essential in theorem 1.5.10?

Question 4: Does theorem 1.5.10 hold for ι_0 or ι_b-equivalence? Does it hold for ι_p^*, ι_0^* or ι_b^*-equivalence?

Before we state our next theorem we first need the following

1.5.11 LEMMA: *Let X and Y be spaces and $\phi: C_p(X) \to C_p(Y)$ a homeomorphism. Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C_p(X)$ such that f_n converges pointwise to a discontinuous function $f \in \mathbb{R}^X$. Suppose $g: Y \to \mathbb{R}$ is an accumulation point of the set $\{\phi(f_n) \mid n \in \mathbb{N}\}$. Then g is not continuous.*

PROOF: Since $\{f_n \mid n \in \mathbb{N}\}$ is closed and discrete in $C_p(X)$ we have $\{\phi(f_n) \mid n \in \mathbb{N}\}$ is closed and discrete in $C_p(Y)$. □

For a space X let $X^{(1)} = \{x \in X : x$ is an accumulation point of $X\}$. We have the following

1.5.12 THEOREM: *Let X and Y be ι_p-equivalent spaces which are both normal and first countable. Then $X^{(1)}$ is countably compact if and only if $Y^{(1)}$ is countably compact.*

PROOF: Suppose $X^{(1)}$ is not countably compact and $Y^{(1)}$ is countably compact. Since $X^{(1)}$ is not sequentially compact, there exists a closed discrete set $F=\{x_n \mid n\in\mathbb{N}\}$ in $X^{(1)}$. For every $n\in\mathbb{N}$ let $\{U_j^n \mid j\in\mathbb{N}\}$ be a decreasing open base at x_n and f_j^n a Urysohn function such that $f_j^n(x_n)=1$ and $f_j^n(X\setminus U_j^n)=0$. Then $f_j^n\to\chi_{x_n}$ pointwise, where χ_{x_n} is the characteristic function of x_n. Notice that χ_{x_n} is discontinuous. Furthermore let ϕ: $C_p(X)\to C_p(Y)$ be a linear homeomorphism and let $g_j^n=\phi(f_j^n)$.

CLAIM: For every $y\in Y$ and $n\in\mathbb{N}$, the set $\{g_j^n(y) \mid j\in\mathbb{N}\}$ is bounded in $\mathbb{R}$.

Suppose not. Then there are $y\in Y$ and $n\in\mathbb{N}$, such that without loss of generality for every $k\in\mathbb{N}$ there is $j_k\in\mathbb{N}$, with $g_{j_k}^n(y)\geq 2^k$. The function $f=\Sigma_{k=1}^{\infty}2^{-k}f_{j_k}^n\in C_p(X)$, so $\phi(f)=\Sigma_{k=1}^{\infty}2^{-k}g_{j_k}^n\in C_p(Y)$. But then we have a contradiction since $\phi(f)(y)=\Sigma_{k=1}^{\infty}2^{-k}g_{j_k}^n(y)=\infty$.

For every $y\in Y$, let A_y be compact in $\mathbb{R}$ such that $\{g_j^n(y) \mid j\in\mathbb{N}\}\subset A_y$. Then $\Pi_{y\in Y}A_y$ is a compact subset of $\mathbb{R}^Y$. Since $\{g_j^n \mid j\in\mathbb{N}\}\subset\Pi_{y\in Y}A_y$, $\{g_j^n \mid j\in\mathbb{N}\}$ has an accumulation point σ_n. By lemma 1.5.11, σ_n is discontinuous, say at y_n. Notice that $y_n\in Y^{(1)}$. Since $Y^{(1)}$ is sequentially compact, without loss of generality we may assume that there is $y\in Y$ such that $y_n\to y$. Let $\{V_n \mid n\in\mathbb{N}\}$ be a decreasing open base at y. Without loss of generality $y_n\in V_n$.

Since Y is first countable, for every $n\in\mathbb{N}$ there is a sequence $(y_k^n)_k$ in V_n such that $y_k^n\to y_n$ and

$$\sigma_n(y_k^n)\not\to\sigma_n(y_n). \qquad (*)$$

Let $K=\bigcup_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}\{y_n, y_k^n\}\cup\{y\}$. Then K is compact. Indeed, let $\mathcal{V}$ be an open cover of K. There is $V\in\mathcal{V}$ with $y\in V$. There is $n_0\in\mathbb{N}$ such that $y\in V_{n_0}\subset V$. Then $\bigcup_{n\geq n_0}\bigcup_{k\in\mathbb{N}}\{y_n, y_k^n\}\cup\{y\}\subset V$. Since $\bigcup_{n<n_0}\bigcup_{k\in\mathbb{N}}\{y_n, y_k^n\}$ is compact, we are done.

Since K is compact, we have by corollary 1.2.15 (b) that $\overline{\operatorname{supp}K}$ is bounded in X. Since F is closed and discrete and X is normal, F is not bounded. This implies that there is $n\in\mathbb{N}$ such that $x_n\notin\overline{\operatorname{supp}K}$. Then there is $j_0\in\mathbb{N}$ such that $U_{j_0}^n\cap\overline{\operatorname{supp}K}=\emptyset$. So for every $z\in K$ and $j\geq j_0$, f_j^n and the zero function on X are equal on $\operatorname{supp}(z)$. Then by corollary 1.4.2, we have that $g_j^n(z)=0$ for every $j\geq j_0$ and $z\in K$. But then for every $k\in\mathbb{N}$ we have that $\sigma_n(y_k^n)=0$ and $\sigma_n(y_n)=0$, which gives a contradiction with (*). This completes the proof of the theorem. □

In section 2.4 we show that the first countability assumption in theorem 1.5.12 is essential. The question remains whether normality is essential.

Our last result in this section deals with the class of metric spaces. Let X be a metric space and $\mathcal{U}$ a family of subsets of X. We define $\operatorname{diam}\mathcal{U}$ to be $\sup\{\operatorname{diam} U : U \in \mathcal{U}\}$. We first need the following

1.5.13 LEMMA: *Let X be a metric space which is not completely metrizable and let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a collection of open covers of X such that for each $n \in \mathbb{N}$, $\operatorname{diam}\mathcal{U}_n < 1/n$. Then there is a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ of natural numbers and for each $n \in \mathbb{N}$, there is $U_n \in \mathcal{U}_{i_n}$ such that $\overline{U}_{n+1} \subset U_n$, and moreover $\bigcap_{n=1}^{\infty} U_n = \emptyset$.*

PROOF: Let $\tilde{X}$ be the completion of X. For each $U \in \mathcal{U}_n$, there is V_U open in $\tilde{X}$ such that $\operatorname{diam} V_U < 3/n$ and $V_U \cap X = U$. Let $V_n = \bigcup\{V_U : U \in \mathcal{U}_n\}$. Then V_n is open in $\tilde{X}$ and $X \subset V_n$. So $V = \bigcap_{n=1}^{\infty} V_n$ is a G_δ-subset of $\tilde{X}$ such that $X \subset V$. Since X is not completely metrizable and V is completely metrizable ([24, Th. 4.3.23]), there is $x \in V \setminus X$.

CLAIM: There is a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ of natural numbers and there are for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_{i_n}$ such that $x \in V_{U_n}$ and $\overline{V}_{U_{n+1}} \subset V_{U_n}$.

Let $i_1 = 1$ and let $U_1 \in \mathcal{U}_1$ be such that $x \in V_{U_1}$. Let $m > 1$ and suppose $i_1, \ldots, i_{m-1}$ and $U_1, \ldots, U_{m-1}$ are found. Let $\delta = d(x, \tilde{X} \setminus V_{U_{m-1}})$ and let $i_m > i_{m-1}$ be such that $1/i_m < \delta/3$. There is $U_m \in \mathcal{U}_{i_m}$ such that $x \in V_{U_m}$. Since $\operatorname{diam}\overline{V}_{U_m} \leq 3/i_m < \delta$ and $x \in V_{U_m}$, $\overline{V}_{U_m} \subset V_{U_{m-1}}$. This proves the claim.

Since $\operatorname{diam} V_{U_m} \to 0$ $(m \to \infty)$, $\bigcap_{m=1}^{\infty} V_{U_m} = \{x\}$. This implies

$$\bigcap_{m=1}^{\infty} U_m = \bigcap_{m=1}^{\infty} V_{U_m} \cap X = \emptyset.$$

Furthermore for $m \in \mathbb{N}$, we have

$$\overline{U}_{m+1} = \overline{V}_{U_{m+1}} \cap X \subset V_{U_m} \cap X = U_m.$$

This proves the lemma. □

Recall from section 1.2 that for metric spaces X and Y and a linear function $\phi: C(X) \to C(Y)$ we defined for $U \subset X$, the set $T_U = \{y \in Y : \operatorname{supp}(y) \cap U \neq \emptyset\}$, and for a family $\mathcal{U}$ of subsets of X the collection $T_{\mathcal{U}} = \{T_U : U \in \mathcal{U}\}$. We now state our last theorem in this section.

1.5.14 THEOREM: *Let X and Y be metric spaces and let $\phi: C_p(X) \to C_p(Y)$ be a continuous linear surjection. If X is completely metrizable, then Y is completely metrizable.*

PROOF: Suppose X is completely metrizable and Y is not completely metrizable. Since X and Y are metric spaces, by lemma 1.2.11 there are locally finite open covers $\mathcal{U}_n$ of X and $\mathcal{V}_n$ of Y, ($n \in \mathbb{N}$), such that

(1) $\operatorname{diam} \mathcal{U}_n < \frac{1}{n}$, $\operatorname{diam} \mathcal{V}_n < \frac{1}{n}$, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, and

(2) each $V \in \mathcal{V}_n$ intersects only finitely many elements of $T_{\mathcal{U}_n}$.

By lemma 1.5.13, we may assume that for each $n \in \mathbb{N}$, there is $V_n \in \mathcal{V}_n$ such that $\bigcap_{n=1}^{\infty} V_n = \emptyset$ and for each $n \in \mathbb{N}$, $\overline{V}_{n+1} \subset V_n$. By (2), for each $n \in \mathbb{N}$, there is a finite subset $\{U_1^n, \ldots, U_{m_n}^n\}$ of $\mathcal{U}_n$ such that

(3) for $U \in \mathcal{U}_n$, $V_n \cap T_U \neq \emptyset$ if and only if $U \in \{U_1^n, \ldots, U_{m_n}^n\}$.

We claim that for each $n \in \mathbb{N}$,

(4) $\bigcup_{j=1}^{m_{n+1}} U_j^{n+1} \subset \bigcup_{j=1}^{m_n} U_j^n$.

Indeed, since $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, there are for each $j \leq m_{n+1}$, $U_j \in \mathcal{U}_n$ such that $U_j^{n+1} \subset U_j$. Since $V_{n+1} \subset V_n$, and $V_{n+1} \cap T_{U_j^{n+1}} \neq \emptyset$, we have $V_n \cap T_{U_j} \neq \emptyset$. So by (3), $U_j \in \{U_1^n, \ldots, U_{m_n}^n\}$. This gives $\bigcup_{j=1}^{m_{n+1}} U_j^{n+1} \subset \bigcup_{j=1}^{m_n} U_j^n$. This proves (4).

Notice that by (3), for every $n \in \mathbb{N}$, $\operatorname{supp} V_n \subset \bigcup_{j=1}^{m_n} U_j^n$. For each $n \in \mathbb{N}$, let $y_n \in V_n$. Then

(5) $\operatorname{supp}(y_n) \subset \bigcup_{j=1}^{m_n} U_j^n$.

Let $K = \overline{\bigcup_{n=1}^{\infty} \operatorname{supp}(y_n)}$. Since K is a closed subset of X, K is complete.

CLAIM: K is compact.

Since K is complete, it remains to prove that K is totally bounded. To this end it suffices to prove that $\bigcup_{n=1}^{\infty} \operatorname{supp}(y_n)$ is totally bounded. Let $\varepsilon > 0$ and let $j \in \mathbb{N}$ be such that $1/j < \varepsilon/2$. For $k \leq m_j$, let $z_k \in U_k^j$. Since $\operatorname{diam} U_k^j < 1/j$, $U_k^j \subset B(z_k, \varepsilon)$. Then by (4) and (5), $\bigcup_{n=j}^{\infty} \operatorname{supp}(y_n) \subset \bigcup_{k=1}^{m_j} B(z_k, \varepsilon)$. Since $\bigcup_{n=1}^{j} \operatorname{supp}(y_n)$ is finite, we are done.

By lemma 1.4.5, $L = \{y \in Y : \mathrm{supp}(y) \subset K\}$ is a closed and bounded subset of Y, hence L is compact. Since $\bigcap_{n=1}^{\infty} \overline{V}_n = \emptyset$, $(y_n)_{n \in \mathbb{N}} \subset L$ is a sequence without convergent subsequence. Contradiction. This proves the theorem. □

1.5.15 COROLLARY: *Complete metrizability is preserved by l_p-equivalence in the class of metric spaces.* □

In [52], Uspenskiĭ proved that for l_p-equivalent spaces X and Y, we have that if X is metric, then Y is a σ-metrizable paracompact space, where σ-metrizable means a countable union of closed metrizable subspaces. In view of theorem 1.5.14, one could conjecture that if X is moreover completely metrizable, then Y is also Čech-complete. This is however not the case. In example 2.4.10 we give two l_p-equivalent spaces X and Y with X countable metric locally compact and Y paracompact σ-metrizable but not Čech-complete.

In general if ϕ: $C_p^*(X) \to C_p^*(Y)$ is a continuous linear surjection, for metric spaces X and Y, the proof of theorem 1.5.14 does not work. By example 1.2.12 we cannot make use of a lemma such as lemma 1.2.11.

Most of the proofs in this section concerning the topology of pointwise convergence depend strongly on corollary 1.2.15 (b). Since we do not have such a result for continuous linear functions between function spaces $C_p^*(X)$, we cannot copy these proofs. The question remains whether the results for l_p-equivalence are also valid for l_p^*-equivalence. Of course there are many more questions to ask. We made a selection in this section and we did not have the intention to be complete.

REMARK: For a recent survey on results obtained for l_p-equivalence we refer to [2]. Theorem 1.5.12 can be found in [5] and theorem 1.5.14 can be found in [7]. As far as we know all other results in this section (except for theorems 1.5.1, 1.5.9 and 1.5.10) are new.

CHAPTER 2

On the l_p and l_0-equivalence of locally compact spaces

The purpose of this chapter is to present isomorphical classifications of function spaces of some locally compact spaces endowed with the topology of pointwise convergence and with the compact open topology. Since ordinals play an important role in the proofs of these classifications, in section 1 we derive some (well-known) properties of ordinals. Other important notions are derivatives of spaces and scatteredness. In section 2 we will give the relevant definitions and present some preliminary results, for example the theorems of Cantor-Bendixson and Sierpiński-Mazurkiewicz. In section 3 we prove some rather general results concerning linear homeomorphisms between certain function spaces.

After these three sections we are in a position to present the first isomorphical classification. In section 4 we present a complete classification of the function spaces $C_p(X)$ for separable metric zero-dimensional compact spaces X. It turns out that this classification is similar to the one Bessaga and Pelczyński gave in [10] for the spaces $C_0(X)$. In section 5 we present a complete isomorphical classification of the function spaces $C_p(X)$, for compact ordinal spaces. This classification is also similar to one for the spaces $C_0(X)$ (viz. the one Kislyakov gave in [34]). In a sense, it is an extension of the classification found in section 4. In section 6 we present a classification of the spaces $C_0(X)$ and $C_p(X)$ for non-compact σ-compact ordinal spaces X. Finally in section 7 we present a complete isomorphical classification of the spaces $C_0(X)$ and $C_p(X)$ for separable metric zero-dimensional locally compact spaces X. This result uses the classifications found in sections 4 and 6.

We already proved that for spaces X and Y having the property that each closed and bounded subset is compact, $C_p(X) \sim C_p(Y)$ implies $C_0(X) \sim C_0(Y)$. It turns out that in each of the classes mentioned above we also have the converse implication. Recall that the converse implication does not hold in general (cf. page 29).

§2.1. Ordinals

Ordinals play an important role in this monograph, in particular prime components. In this section we will present some facts about ordinals, we will give the definitions of initial, regular and singular ordinals and of prime components, and we prove some (well-known) results. Most of these can be found in [48] and [35]. For definitions of ordinals, cardinals and related topics which are not defined or proved in this section, we refer to [48], [35] and [32]. In this section, every greek letter denotes an ordinal, and finite ordinals will be denoted by n or m occasionally.

We begin with stating some basic properties of addition, multiplication and exponentation of ordinals. Recall that addition of ordinals is associative but not commutative. If α and β are ordinals, then $\alpha+\beta\geq\beta$ and if $\alpha>0$, then $\beta+\alpha>\beta$. Observe that not always $\alpha+\beta>\beta$ because $1+\omega=\omega$. Another important property of addition is the following: If $\alpha\geq\beta$, then there is exactly one ordinal γ such that $\alpha=\beta+\gamma$. We denote this number γ by $\alpha-\beta$. With these properties one can easily derive the following

2.1.1 PROPOSITION: *Let α, β, γ and δ be ordinals. Then*

(*a*) $\beta<\gamma$ *implies* $\alpha+\beta<\alpha+\gamma$,

(*b*) $\beta<\gamma$ *implies* $\beta+\alpha\leq\gamma+\alpha$, *and*

(*c*) $\alpha<\gamma$ *and* $\beta<\delta$ *implies* $\alpha+\beta<\gamma+\delta$. □

Like addition, multiplication is associative but not commutative (for example $2\cdot\omega=\omega\neq\omega\cdot2$). Now let α, β, and γ be ordinals. Then $\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$, but in general $(\beta+\gamma)\cdot\alpha\neq\beta\cdot\alpha+\gamma\cdot\alpha$ (for example $(1+1)\cdot\omega=\omega\neq\omega\cdot2=1\cdot\omega+1\cdot\omega$). One can now easily deduce that if $\alpha>0$ and $\beta>\gamma$, then $\alpha\cdot\beta>\alpha\cdot\gamma$, however, if $\beta>\gamma$ then $\beta\cdot\alpha\geq\gamma\cdot\alpha$ (notice that $2\cdot\omega=1\cdot\omega$).

We also have the following important

2.1.2 PROPOSITION: *Let α and β be ordinals. If $\alpha>0$ then there are ordinals μ and ν such that $\beta=\alpha\cdot\mu+\nu$ with $\nu<\alpha$.*

2.1.3 COROLLARY: *Let α, β and γ be ordinals such that $\beta<\alpha\cdot\gamma$. If $\alpha>0$ then there are ordinals μ and ν such that $\beta=\alpha\cdot\mu+\nu$ with $\mu<\gamma$ and $\nu<\alpha$.*

PROOF: By proposition 2.1.2 there are ordinals μ and ν such that $\beta=\alpha\cdot\mu+\nu$ with $\nu<\alpha$. By proposition 2.1.1 (a) $\beta\geq\alpha\cdot\mu$. So if $\mu\geq\gamma$, then $\beta\geq\alpha\cdot\mu\geq\alpha\cdot\gamma$, which is a con-

tradiction. Hence $\mu < \gamma$. □

Exponentation of ordinals α and β is defined by transfinite induction as follows:

a) If $\beta = 0$ then $\alpha^\beta = 1$,
b) if β is a successor, say $\beta = \gamma + 1$, then $\alpha^\beta = \alpha^\gamma \cdot \alpha$,
c) if β is a limit ordinal, then $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$.

With this definition one can easily prove the following

2.1.4 PROPOSITION: *Let α, β and γ be ordinals. Then*
(a) if $\alpha > 1$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$,
(b) if $\alpha > 1$ then $\alpha^\beta \geq \beta$, and
(c) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. □

The following lemma will be used in section 2.5.

2.1.5 LEMMA: *Let $\alpha > 1$ and $\beta \geq 1$ be ordinals. Then there are $\gamma \leq \beta$, $1 \leq \lambda < \alpha$ and $\delta < \alpha^\gamma$ such that $\beta = \alpha^\gamma \cdot \lambda + \delta$.*

PROOF: By proposition 2.1.4 (a) and (b), $\beta \leq \alpha^\beta < \alpha^{\beta+1}$, so the set $A = \{\nu : \alpha^\nu > \beta\}$ is non-empty. Let $\mu = \min A$. Notice that $1 \leq \mu \leq \beta + 1$. If μ is a limit ordinal, then $\beta < \alpha^\mu = \sup\{\alpha^\nu : \nu < \mu\}$ implies there is $\nu < \mu$ with $\beta < \alpha^\nu$. This is a contradiction, so $\mu = \gamma + 1$ for some γ. Since $\mu \leq \beta + 1$, $\gamma \leq \beta$.

Since $\beta < \alpha^\gamma \cdot \alpha$, by corollary 2.1.3, there is $\delta < \alpha^\gamma$ and $\lambda < \alpha$ such that $\beta = \alpha^\gamma \cdot \lambda + \delta$. If $\lambda = 0$, then $\beta = \delta < \alpha^\gamma$ which is impossible, so $\lambda \geq 1$. □

Let α be an ordinal. By $\overline{\alpha}$ we denote the cardinality of α (i.e., $\overline{\alpha} = |W(\alpha)|$) and we call $\overline{\alpha}$ the ***power*** of α. An ordinal $\phi \geq \omega$ is called an ***initial ordinal*** if ϕ is the smallest ordinal β such that $\overline{\beta} = \overline{\phi}$, i.e., $\gamma < \phi$ implies $\overline{\gamma} < \overline{\phi}$. To every initial ordinal ϕ we assign the ***index*** $i(\phi)$ of ϕ as the ordertype of the set $P(\phi) = \{\psi < \phi : \psi \text{ is initial}\}$. For example $i(\omega) = 0$ and $i(\omega_1) = 1$. Notice that for every initial ordinal ϕ, $i(\phi) \leq \phi$.

The following theorem easily follows from the above definitions.

2.1.6 THEOREM ([35, Th. 3, p273]): *If ψ and ϕ are initial ordinals with $\psi < \phi$, then $i(\psi) < i(\phi)$.* □

As a direct consequence of this theorem we remark that to distinct initial ordinals

correspond distinct indices. So we may denote the initial ordinal ϕ with index α by ω_α. Since $i(\phi) \leq \phi$, it follows that $\alpha \leq \omega_\alpha$. The following theorem states that ω_α is defined for every ordinal α.

2.1.7 THEOREM ([35, Th. 5, p273]): *Every ordinal is the index of some initial ordinal.*

An ***α-sequence*** is a function ϕ with domain $W(\alpha)$, whose values are ordinals. If $\gamma < \beta < \alpha$ implies $\phi(\gamma) < \phi(\beta)$, then ϕ is an ***increasing α-sequence***. The ***limit $\lim_{\xi<\alpha}\phi(\xi)$ of an increasing α-sequence*** ϕ is the ordinal $\sup\{\phi(\xi) : \xi < \alpha\}$. We say that an ordinal λ is ***cofinal*** with a limit ordinal α, if λ is the limit of an increasing α-sequence ϕ, i.e., $\lambda = \lim_{\xi<\alpha}\phi(\xi)$.

2.1.8 THEOREM ([35, Th. 8, p274]): *If λ is a limit ordinal, then $\omega_\lambda = \lim_{\xi<\lambda}\omega_\xi$.*

2.1.9 THEOREM ([35, Th. 10, p274]): *Let $\lambda > 0$ be a limit ordinal. The smallest ordinal α such that λ is cofinal with α is an initial ordinal.*

If we now define for every limit ordinal λ, the ordinal $cf(\lambda)$ as the smallest α such that λ is cofinal with α, then by theorem 2.1.9, $cf(\lambda)$ is an initial ordinal. For example $cf(\omega) = \omega$, $cf(\omega_1) = \omega_1$, $cf(\omega_\omega) = \omega$ and $cf(\omega_{\omega_1}) = \omega_1$. Since $\lambda = \lim_{\beta<\lambda}\beta$, λ is cofinal with itself, hence $cf(\lambda) \leq \lambda$

This observation leads us to the following definition. If $cf(\omega_\alpha) = \omega_\alpha$, we call ω_α a ***regular initial ordinal*** or shortly regular. Otherwise it is called a ***singular initial ordinal*** or shortly singular. For example, ω and ω_1 are regular and ω_ω is singular. This is standard terminology of course.

We are going to prove that if ω_γ is singular, then γ is a limit ordinal (cf. theorem 2.1.10). For that we first recall the following well-known fact: If $\mathfrak{m}$ and $\mathfrak{n}$ are cardinals which are not both finite, then $\mathfrak{m} + \mathfrak{n} = \max(\mathfrak{m}, \mathfrak{n}) = \mathfrak{m} \cdot \mathfrak{n}$. This allows us to derive some important corollaries.

2.1.10 THEOREM ([35, Th. 9, p278]): *If α is a successor, then ω_α is regular.*

PROOF: Let $\alpha = \beta + 1$ and $\omega_\gamma = cf(\omega_\alpha)$. That means there is an increasing ω_γ-sequence ϕ such that $\lim_{\xi<\omega_\gamma}\phi(\xi) = \omega_\alpha$. But then $W(\omega_\alpha) \subset \bigcup_{\xi<\omega_\gamma} W(\phi(\xi))$. Notice that $|W(\phi(\xi))| \leq \overline{\omega_\beta}$. So we have by the above remark and theorem 2.1.6

$$\overline{\omega_\alpha} = |W(\omega_\alpha)| \leq |\bigcup_{\xi<\omega_\gamma} W(\phi(\xi))| \leq \Sigma_{\xi<\omega_\gamma} |W(\phi(\xi))|$$
$$\leq \Sigma_{\xi<\omega_\gamma} \overline{\omega_\beta} = \overline{\omega_\gamma}\cdot\overline{\omega_\beta} = \overline{\omega_{\max(\beta,\gamma)}}.$$

It follows by theorem 2.1.6 that $\alpha \leq \max(\beta, \gamma)$, so $\alpha \leq \gamma$. Since $\omega_\gamma = cf(\omega_\alpha) \leq \omega_\alpha$, by theorem 2.1.6 $\gamma \leq \alpha$ and thus $\gamma = \alpha$. □

2.1.11 COROLLARY: *If ω_α is singular, then α is a limit ordinal.* □

The next two results are going to be used in section 2.5.

2.1.12 PROPOSITION: $\alpha\cdot\omega_\beta = \omega_\beta$ *for every* $0 < \alpha < \omega_\beta$.

PROOF: By the remark on page 52, $\overline{\alpha\cdot\omega_\beta} = \overline{\omega_\beta}$, so we are done if we prove that $\alpha\cdot\omega_\beta$ is initial. To this end, let $\delta < \alpha\cdot\omega_\beta$. By corollary 2.1.3 there are $\mu < \omega_\beta$ and $\nu < \alpha$ such that $\delta = \alpha\cdot\mu + \nu$. So $\overline{\delta} = \overline{\alpha}\cdot\overline{\mu} + \overline{\nu} = \max\{\overline{\alpha}, \overline{\mu}, \overline{\nu}\}$. Since $\alpha, \mu, \nu < \omega_\beta$ it follows that $\overline{\delta} < \overline{\omega_\beta}$, and therefore $\alpha\cdot\omega_\beta$ is initial. □

Notice that proposition 2.1.12 is not true for $\alpha = \omega_\beta$. For example, $\omega^2 \neq \omega$.

2.1.13 PROPOSITION: *Let $\alpha \geq \omega$ be an ordinal. If $\overline{\gamma} \leq \overline{\alpha}$ then $\overline{\alpha^\gamma} = \overline{\alpha}$.*

PROOF: We prove this by induction on γ. If $\gamma = 1$ then it is a triviality, so let $\gamma > 1$ and suppose the proposition is true for every $\delta < \gamma$.

Case 1: γ is a successor, say $\gamma = \delta + 1$.

Then by the the remark on page 52, $\overline{\alpha^\gamma} = \overline{\alpha^\delta\cdot\alpha} = \overline{\alpha}\cdot\overline{\alpha} = \overline{\alpha}$.

Case 2: γ is a limit ordinal.

Then

$$\overline{\alpha^\gamma} = \overline{\lim_{\delta<\gamma}\alpha^\delta} = |\bigcup_{\delta<\gamma} W(\alpha^\delta)| \leq \Sigma_{\delta<\gamma} |W(\alpha^\delta)|$$
$$= \Sigma_{\delta<\gamma} \overline{\alpha^\delta} = \overline{\gamma}\cdot\overline{\alpha} = \overline{\alpha}$$

(by the fact that $\overline{\gamma} \leq \overline{\alpha}$). □

Notice that $\overline{\alpha^\gamma}$ need not be equal to $\overline{\alpha}^{\overline{\gamma}}$. For example, $\overline{\omega}^{\overline{\omega}} > \overline{\omega}$, but $\overline{\omega^\omega} = \overline{\omega}$.

2.1.14 EXAMPLE: There is a singular ordinal ω_α such that $\alpha = \omega_\alpha$. Indeed, define the sequence (β_n) of ordinals inductively as follows: $\beta_0 = 0$ and $\beta_{n+1} = \omega_{\beta_n}$. Let $\alpha = \sup\{\beta_n : n \in \mathbb{N}\}$. Then $\omega_\alpha = \lim_{n<\omega}\beta_n$, hence $cf(\omega_\alpha) = \omega$. Furthermore $\alpha = \omega_\alpha$.

We now come to the following definition. An ordinal ρ is a ***prime component*** if it satisfies the following condition:

If $\rho = \beta + \gamma$ for some ordinals β and γ, then $\gamma = 0$ or $\gamma = \rho$.

Examples of prime components are ω and ω_1. The ordinals 3, $\omega + 1$ and $\omega \cdot 2$ are examples of ordinals which are not prime components. Furthermore, 0 and 1 are the only finite prime components. Since prime components play a very important role in this monograph and since their properties are not well-known, we prove all the properties that we need.

2.1.15 THEOREM ([48, Th. 1, p279]): *An ordinal ρ is a prime component if and only if for every ordinal $\beta < \rho$, $\rho = \beta + \rho$.*

PROOF: Suppose that the ordinal ρ is a prime component, and let $\beta < \rho$. There is an ordinal γ such that $\rho = \beta + \gamma$ (see page 50). From the definition of prime component it follows that $\gamma = 0$ or $\gamma = \rho$. Since $\beta < \rho$, we have $\gamma = \rho$.

Now suppose that ρ satisfies the condition mentioned and that $\rho = \beta + \gamma$. Assume $\gamma > 0$. By proposition 2.1.1 (a) it follows that $\beta < \beta + \gamma = \rho$. But then $\rho = \beta + \rho$ and we may conclude that $\gamma = \rho$ and so ρ is a prime component. □

The next theorem plays an important role in section 2.6.

2.1.16 THEOREM ([48, Th. 2, p278]): *For every ordinal $\alpha > 0$, there is an ordinal β and a prime component $\rho > 0$ such that $\alpha = \beta + \rho$, where $\beta = 0$ or $\beta \geq \rho$.*

PROOF: Let $\alpha > 0$ be an ordinal and $A = \{\tau > 0$: there is β such that $\alpha = \beta + \tau\}$. Notice that $\tau \in A$ implies $\tau \leq \alpha$ (because by proposition 2.1.1 (b) $\alpha = \beta + \tau \geq \tau$). Let $\rho = \min A$ (which exists because A is a non-empty subset of $W(\alpha)$ and the last set is well-ordered) and pick β such that $\alpha = \beta + \rho$.

We prove that ρ is a prime component. Indeed, suppose that $\rho = \mu + \nu$, with $\nu > 0$. Then $\alpha = \beta + (\mu + \nu) = (\beta + \mu) + \nu$, so $\nu \in A$ and thus $\nu \geq \rho$. Since $\mu + \nu \geq \nu$ (proposition 2.1.1 (b)) we have $\nu \leq \rho$, so $\nu = \rho$ and hence ρ is a prime component.

Finally, if $\beta < \rho$, then by theorem 2.1.15, $\alpha = \beta + \rho = \rho$, so in that case we can choose $\beta = 0$. □

2.1.17 LEMMA ([48, lemma p282]): *Let P be a set of prime components. Then $\sup P$ is a prime component.*

PROOF: Let $\beta = \sup P$. We prove that β is a prime component. Let $\gamma < \beta$. Then there is a prime component $\rho \in P$ with $\gamma < \rho < \beta$. Let $\delta = \beta - \rho$ (see page 50). Then by theorem 2.1.15,

$$\gamma + \beta = \gamma + (\rho + \delta) = (\gamma + \rho) + \delta = \rho + \delta = \beta.$$

So by theorem 2.1.15, β is a prime component. □

By applying this lemma to the set $P = \{\rho \leq \alpha : \rho \text{ is a prime component}\}$, we get the following important

2.1.18 COROLLARY ([48, lemma p282]): *Let α be an ordinal. Then there is a largest prime component which is less than or equal to α.* □

In the sequel we denote the largest prime component which is less than or equal to a given ordinal α by α'.

2.1.19 LEMMA ([48, Cor. p305]): *If $\alpha > 0$ is an ordinal, then $\alpha\omega$ is the smallest prime component larger than α.*

PROOF: We first prove that $\alpha\omega$ is a prime component. So suppose $\alpha\omega = \mu + \nu$ with $\nu \neq 0$ and $\nu \neq \alpha\omega$. By proposition 2.1.1 (a) and (b) it then follows that $\mu, \nu < \alpha\omega$. By corollary 2.1.3 there are ordinals m, n, γ_1 and γ_2 such that $\mu = \alpha m + \gamma_1$, $\nu = \alpha n + \gamma_2$, m, $n < \omega$, and $\gamma_1, \gamma_2 < \alpha$.

From proposition 2.1.1 (a) it follows that $\mu < \alpha m + \alpha$ and $\nu < \alpha n + \alpha$. Now with proposition 2.1.1 (c) it follows that $\alpha\omega = \mu + \nu < \alpha m + \alpha + \alpha n + \alpha$, so by the remark on page 50, $\alpha\omega < \alpha(m + n + 2) < \alpha\omega$. This is a contradiction and we conclude that $\alpha\omega$ is a prime component.

Now suppose that there is a prime component ρ such that $\alpha < \rho < \alpha\omega$. By corollary 2.1.3 there are ordinals n and γ with n finite and $\gamma < \alpha$ such that $\rho = \alpha n + \gamma$. Since ρ is a prime component, $\gamma = 0$ and it follows that $\rho = \alpha n$. Since $\alpha < \rho$, $n > 1$, so $\rho = \alpha(n-1) + \alpha$. Since ρ is a prime component and $0 < \alpha < \rho$ we arrived at a contradiction. □

2.1.20 COROLLARY: *Let α be an ordinal. Then there is $n \in \mathbb{N}$ and $\gamma < \alpha'$ such that $\alpha = \alpha' \cdot n + \gamma$.*

PROOF: Since $\alpha' \cdot \omega$ is the smallest prime component larger than α' (lemma 2.1.19),

$\alpha' \leq \alpha < \alpha'\omega$. So by corollary 2.1.3 there are ordinals $n < \omega$ and $\gamma < \alpha'$ such that $\alpha = \alpha' n + \gamma$. Since $\alpha \geq \alpha'$ we have $n \neq 0$, hence $n \in \mathbb{N}$.

The following theorem will often be used in this monograph.

2.1.21 THEOREM ([48, Th. 1, p320]): *An ordinal $\rho > 0$ is a prime component if and only if there is an ordinal μ such that $\rho = \omega^\mu$.*

PROOF: Let $\rho > 0$ be a prime component.

CLAIM: There is α such that $\omega^\alpha \leq \rho < \omega^{\alpha+1}$.

Indeed, by lemma 2.1.5 there are $\alpha \leq \rho$, $1 \leq \lambda < \omega$ and $\delta < \omega^\alpha$ such that $\rho = \omega^\alpha \cdot \lambda + \delta$. Thus $\omega^\alpha \leq \rho < \omega^{\alpha+1}$, and the claim is proved.

Since ρ is a prime component and since the smallest prime component larger than ω^α is $\omega^\alpha \cdot \omega = \omega^{\alpha+1}$ (lemma 2.1.19), it follows by the claim that $\rho = \omega^\alpha$

For the converse implication suppose there is ν such that ω^ν is not a prime component. Let μ be the smallest among them. Suppose that $\mu = \nu + 1$. Then by lemma 2.1.19 $\omega^\mu = \omega^\nu \cdot \omega$ is a prime component, which is not true, hence μ is a limit ordinal. But then $\omega^\mu = \sup\{\omega^\nu : \nu < \mu\}$. Since for $\nu < \mu$, ω^ν is a prime component, by lemma 2.1.17, ω^μ is a prime component. Contradiction. □

The next lemma will be used in section 2.4.

2.1.22 LEMMA: *Let α and β be ordinals such that $\alpha \geq \omega$ and $\alpha \leq \beta < \alpha^\omega$. Then $\alpha' \leq \beta < (\alpha')^\omega$.*

PROOF: Since $\alpha \leq \beta < \alpha^\omega$, there is $n \in \mathbb{N}$ such that $\alpha \leq \beta < \alpha^n$. Furthermore by theorem 2.1.21 $(\alpha')^2$ is a prime component, from which we may conclude that $\alpha' \leq \alpha < (\alpha')^2$. Since $\alpha < (\alpha')^2$, it is easily seen that $\alpha^n < (\alpha')^{2n}$ (by induction and the remarks on page 50). Whence (by proposition 2.1.4 (a)) $\alpha' \leq \beta < (\alpha')^{2n} < (\alpha')^\omega$. □

2.1.23 THEOREM ([35, Th. 9, p274]): *Every initial ordinal is a prime component.*

PROOF: Let ϕ be an initial ordinal. By corollary 2.1.20 there are $n \in \mathbb{N}$ and $\gamma < \phi'$ such that $\phi = \phi' \cdot n + \gamma$. So by proposition 2.1.1 (a), $\phi' \leq \phi < \phi' \cdot (n+1)$. By theorem 2.1.21, there is an ordinal μ such that $\phi' = \omega^\mu$. Notice that $\mu \geq 1$ (since $\phi \geq \omega$), so there is δ, such that $\mu = 1 + \delta$. By proposition 2.1.4 (c), $\omega^{1+\delta} = \omega \cdot \omega^\delta$ and therefore

$$\overline{\phi'\cdot(n+1)} = \overline{\omega^{1+\delta}\cdot(n+1)} = \overline{(n+1)\cdot\omega\cdot\omega^{\delta}} = \overline{\omega^{1+\delta}} = \overline{\phi'}.$$

It follows that $\bar{\phi} = \overline{\phi'}$ and since ϕ is initial, $\phi' = \phi$. □

2.1.24 COROLLARY: *If ω^{τ} is initial then τ is a prime component.*

PROOF: Let $\nu < \tau$. Then $\omega^{\nu} < \omega^{\tau}$ (proposition 2.1.4 (a)), so $\omega^{\nu}\cdot\omega^{\tau} = \omega^{\tau}$ (proposition 2.1.12). So by proposition 2.1.4 (c), $\nu + \tau = \tau$, hence by theorem 2.1.15 τ is a prime component. □

Finally we remark that the proofs of proposition 2.1.12, corollary 2.1.13 and proposition 2.1.24 are due to us and were included because we could not find a reference.

§2.2. Derivatives and scattered spaces

In this section we briefly discuss some properties of derivatives of sets and scatteredness. Furthermore we formulate the well-known theorems of Cantor-Bendixson and Sierpiński-Mazurkiewicz and we present some results which we need in section 2.3 and in chapter 4.

Let X be a topological space and let $A \subset X$. The ***derived set A^d of A in X*** is defined to be the set of all $x \in X$ satisfying the condition that for every neighborhood U of x (in X), $U \cap A \setminus \{x\} \neq \emptyset$ (i.e., the set of all accumulation points of A in X). Notice that not necessarily $A^d \subset A$: for example let $X = \mathbb{R}$ and $A = (0, 1)$. It is well-known that $A^d \subset \overline{A}$ and that A^d is closed in X. Now for every ordinal α we define $X^{(\alpha)}$, the ***α-th derivative***, by transfinite induction as follows:

a) $X^{(0)} = X$,
b) if α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{(\beta)})^d$,
c) if α is a limit ordinal then $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$.

Notice that $X^{(1)}$ is the derived set of X in X. Furthermore we get $X^{(1)}$ from X by "throwing away" all isolated points of X. The above is standard notation of course.

2.2.1 REMARK: If A is a subspace of a topological space X, then $A^{(1)}$ is the derived set of A in A (so $A^{(1)} \subset A$) and we put A^d the derived set of A in X. Since not

necessarily $A^d \subset A$ it follows that in general $A^{(1)} \neq A^d$. We claim that $A^{(1)} = A^d \cap A$. In particular, if A is closed, then $A^{(1)} = A^d$.

Indeed, let $x \in A^{(1)}$. Since $A^{(1)} \subset A$, $x \in A$. Let U be a neighborhood of x in X. Then $U \cap A$ is a neighborhood of x in A. Since $x \in A^{(1)}$, $\emptyset \neq U \cap A \cap A \setminus \{x\} = U \cap A \setminus \{x\}$ and hence $x \in A^d$. For the reverse inclusion let $x \in A \cap A^d$ and let U be a neighborhood of x in A. Let V be a neighborhood of x in X such that $V \cap A = U$. Since $x \in A^d$, $\emptyset \neq V \cap A \setminus \{x\} = U \cap A \setminus \{x\}$, so $x \in A^{(1)}$.

2.2.2 PROPOSITION: *Let X be a space. Then for every ordinal α and β with $\alpha \leq \beta$*

(a) $X^{(\alpha)}$ is closed in X, and

(b) $X^{(\beta)} \subset X^{(\alpha)}$.

PROOF: We prove (a) by transfinite induction. For $\alpha = 0$ it is trivial. So let $\alpha > 0$ and suppose (a) is proved for every $\beta < \alpha$. If α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{(\beta)})^d$ is closed in X. If α is a limit ordinal then by the inductive hypothesis, $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$ is closed in X, and (a) is proved.

We prove (b) by transfinite induction an β. Notice that (b) is obviously true if $\beta = \alpha$, so suppose that $\beta > \alpha$ and (b) is proved for every γ with $\alpha \leq \gamma < \beta$. If $\beta = \gamma + 1$ for some ordinal γ, then by the inductive hypothesis $X^{(\beta)} = (X^{(\gamma)})^d \subset \overline{X^{(\gamma)}} \subset \overline{X^{(\alpha)}}$. Since by (a), $X^{(\alpha)}$ is closed, we have the desired result.

If β is a limit ordinal, then $X^{(\beta)} = \bigcap_{\gamma < \beta} X^{(\gamma)} \subset X^{(\alpha)}$. □

From remark 2.2.1 and proposition 2.2.2 (a) we easily get the following

2.2.3 COROLLARY: *Let X be a space and α an ordinal. Then $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$.* □

2.2.4 PROPOSITION: *Let X be a space and A a subspace of X. Then for each ordinal α,*

(a) $A^{(\alpha)} \subset X^{(\alpha)}$, and

(b) if A is open then $A^{(\alpha)} = A \cap X^{(\alpha)}$.

PROOF: We prove this proposition by transfinite induction on α. If $\alpha = 0$, the proposition is obviously true, so suppose that $\alpha > 0$ and that the proposition is proved for every $\beta < \alpha$. First suppose that α is a successor, say $\alpha = \beta + 1$.

For (a), since by the inductive hypothesis $A^{(\beta)} \subset X^{(\beta)}$, by [24, Th. 1.3.4 (ii)] $A^{(\alpha)} = (A^{(\beta)})^d \subset (X^{(\beta)})^d = X^{(\alpha)}$.

For (b) let $x \in A \cap X^{(\alpha)}$ and let U be a neighborhood of x in A. Since A is open, U is a neighborhood of x in X, hence $U \cap A \cap X^{(\beta)} \setminus \{x\} = U \cap X^{(\beta)} \setminus \{x\} \neq \emptyset$ (because $x \in X^{(\alpha)}$). So by the inductive hypothesis $\emptyset \neq U \cap A \cap X^{(\beta)} \setminus \{x\} \subset U \cap A^{(\beta)} \setminus \{x\}$, hence $x \in A^{(\alpha)}$. The reverse inclusion follows directly from (a).

If α is a limit ordinal (a) and (b) easily follow from the inductive hypothesis and the definitions of $A^{(\alpha)}$ and $X^{(\alpha)}$. □

2.2.5 PROPOSITION: *Let α be an ordinal and $X = [1, \omega^\alpha]$. Then $X^{(\alpha)} = \{\omega^\alpha\}$.*

PROOF: If $\alpha = 0$ the proposition is obviously true, so suppose that $\alpha > 0$ and that the proposition has been proved for every $\beta < \alpha$. First suppose that α is a successor, say $\alpha = \beta + 1$. If for $i \in \mathbb{N} \cup \{0\}$, $X_i = [\omega^\beta \cdot i + 1, \omega^\beta \cdot (i+1)]$, then

$$X = \bigcup_{i=0}^{\infty} X_i \cup \{\omega^\alpha\},$$

X_i is open in X and $X_i \approx [1, \omega^\beta]$, so by the inductive hypothesis $(X_i)^{(\beta)} = \{\omega^\beta \cdot (i+1)\}$. By proposition 2.2.4 (b), $X_i \cap X^{(\beta)} = (X_i)^{(\beta)}$. Since ω^α is an accumulation point of $\{\omega^\beta \cdot (i+1) : i \geq 0\}$, we conclude that $X^{(\beta)} = \{\omega^\beta \cdot (i+1) : i \geq 0\} \cup \{\omega^\alpha\}$ and so $X^{(\alpha)} = \{\omega^\alpha\}$.

Now suppose α is a limit ordinal. Fix $\beta < \alpha$ and let $\beta \leq \gamma < \alpha$. Let $A = [1, \omega^\gamma]$. By the inductive hypothesis and propositions 2.2.4 (a) and 2.2.2 (a), $\{\omega^\gamma\} = A^{(\gamma)} \subset X^{(\gamma)} \subset X^{(\beta)}$. Since ω^α is an accumulation point of $\{\omega^\gamma : \beta \leq \gamma < \alpha\}$, it follows that $\omega^\alpha \in X^{(\beta)} \subset X^{(\alpha)}$. For the reverse inclusion let $\xi \in X \setminus \{\omega^\alpha\}$. Then there is $\beta < \alpha$ such that $\xi < \omega^\beta$. Then $\xi \in [1, \omega^\beta]$, which is open in X, so by proposition 2.2.4 (b) and the inductive hypothesis we have $\{\omega^\beta\} = [1, \omega^\beta]^{(\beta)} = [1, \omega^\beta] \cap X^{(\beta)}$. So $\xi \notin X^{(\beta)}$ and hence $\xi \notin X^{(\alpha)}$. □

Let A be a subspace of X. A is ***dense in itself*** if $A \subset A^d$ or equivalently $A = A^{(1)}$. This means that A contains no isolated points. A is ***scattered*** if A contains no dense in itself subsets, i.e., every subset of A contains isolated points. Again this is standard terminology.

After these definitions we can state the theorem of Cantor-Bendixson (cf. [24, p85] or [47, p148]).

2.2.6 THEOREM (Cantor-Bendixson): *Let X be a topological space. Then there exists an ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$. For this α, $X^{(\alpha)}$ is closed and dense in itself and $S = X \setminus X^{(\alpha)}$ is scattered. In particular, X is scattered if and only if $X^{(\alpha)} = \emptyset$.*

Furthermore, if X is second countable then S is countable.

Let X be a scattered space. By theorem 2.2.6, there is an ordinal α such that $X^{(\alpha)} = \emptyset$. Now the ***scattered height*** $\kappa(X)$ of X is defined to be the smallest ordinal α such that $X^{(\alpha)} = \emptyset$. It is easy to see that if X is compact and scattered, then $\kappa(X)$ is a successor, say $\alpha+1$ and $X^{(\alpha)}$ contains only finitely many points. If X is second countable and scattered then $\kappa(X)$ is countable. Notice that by proposition 2.2.5, $\kappa([1, \omega^\alpha]) = \alpha + 1$.

2.2.7 REMARK: Every countable compact Hausdorff space is scattered. Indeed, let X be a countable compact Hausdorff space. Then X is second countable ([24, th 3.1.21]) and regular, hence X is metrizable ([24, th 4.2.9]). So since X is countable, it is also zero-dimensional. Now suppose X is not scattered. Then by theorem 2.2.6 there is $P \subset X$ closed, non-empty and dense in itself. Then P is separable metric zero-dimensional and compact without isolated points. Thus $X \approx C$ [15], so P is uncountable, which is a contradiction.

We can now formulate the theorem of Sierpiński-Mazurkiewicz (cf. [47, p155] or [36]).

2.2.8 THEOREM (Sierpiński-Mazurkiewicz): *Let X be a countable compact Hausdorff space. If $\kappa(X) = \alpha + 1$ and $X^{(\alpha)}$ contains m points (m finite), then $X \approx [1, \omega^\alpha \cdot m]$.*

Notice that by proposition 2.2.5 it easily follows that if $X = [1, \omega^\alpha \cdot m]$, with α countable and $m \in \mathbb{N}$, then $\kappa(X) = \alpha + 1$ and $X^{(\alpha)} = \{\omega^\alpha \cdot 1, \ldots, \omega^\alpha \cdot m\}$.

Now we will prove some simple results, which we will need in section 2.3. Let X be a topological space and A a nonempty closed subset of X. Let X/A be the quotient space obtained from X by identifying A to a single point, say ∞ and let $p: X \to X/A$ be the quotient map. Notice that p is closed.

The next lemma gives some results on the derivatives of X/A in terms of the derivatives of X, if A is of a special form.

2.2.9 LEMMA: *Let X be a space, α an ordinal such that $A = X^{(\alpha)} \neq \emptyset$, and $Y = X/A$. Then*

(a) for every $\beta \leq \alpha$, and $p(X^{(\beta)}) = Y^{(\beta)}$, and
(b) $Y^{(\alpha)} = \{\infty\}$.

PROOF: We first prove (a). Notice that by propositions 2.2.4 (b) and 2.2.2 (b),

$$X^{(\beta)} = (X^{(\beta)} \cap (X \setminus A)) \cup (X^{(\beta)} \cap A) = (X \setminus A)^{(\beta)} \cup A,$$

for every $\beta \leq \alpha$. In the same way $Y^{(\beta)} = (Y \setminus \{\infty\})^{(\beta)} \cup (Y^{(\beta)} \cap \{\infty\})$. Since $p \restriction X \setminus A : X \setminus A \to Y \setminus \{\infty\}$ is a homeomorphism, $p((X \setminus A)^{(\beta)}) = (Y \setminus \{\infty\})^{(\beta)}$ for every $\beta \leq \alpha$.

We prove (a) by induction on β. For $\beta = 0$ this is a triviality, so let $0 < \beta \leq \alpha$ and assume it is true for every $\gamma < \beta$.

Case 1: β is a successor, say $\beta = \gamma + 1$.

Suppose $\infty \notin Y^{(\beta)}$. By the inductive hypothesis and since $A \subset X^{(\gamma)}$, $\infty \in Y^{(\gamma)}$, thus $\{\infty\}$ is open in $Y^{(\gamma)}$. But then A is open in $X^{(\gamma)}$, so by proposition 2.2.4 (b) and corollary 2.2.3

$$A^{(1)} = A \cap (X^{(\gamma)})^{(1)} = A \cap X^{(\beta)} = A,$$

which gives a contradiction. Hence $\infty \in Y^{(\beta)}$, so by the above remarks

$$\begin{aligned} p(X^{(\beta)}) &= p((X \setminus A)^{(\beta)} \cup A) \\ &= (Y \setminus \{\infty\})^{(\beta)} \cup \{\infty\} \\ &= (Y \setminus \{\infty\})^{(\beta)} \cup (Y^{(\beta)} \cap \{\infty\}) \\ &= Y^{(\beta)}. \end{aligned}$$

Case 2: β is a limit ordinal.

Then

$$Y^{(\beta)} = \bigcap_{\gamma<\beta} Y^{(\gamma)} = \bigcap_{\gamma<\beta} p(X^{(\gamma)}) = p\left(\bigcap_{\gamma<\beta} X^{(\gamma)}\right) = p(X^{(\beta)}).$$

This finishes the proof of (a).

By (a) we have $Y^{(\alpha)} = p(X^{(\alpha)}) = p(A) = \{\infty\}$, which proves (b). □

With this lemma we can give a classification of X/A, for X a countable compact space.

2.2.10 COROLLARY: *Let X be a countable compact space and let $A = X^{(\alpha)}$ for some $\alpha < \kappa(X)$. Then $X/A \approx [1, \omega^\alpha]$. In other words, if $X = [1, \omega^\alpha \cdot n]$ for certain $n \in \mathbb{N}$ and $\alpha < \omega_1$ (so $A = X^{(\alpha)} = \{\omega^\alpha \cdot 1, \ldots, \omega^\alpha \cdot n\}$) then $X/A \approx [1, \omega^\alpha]$.*

PROOF: The first part follows from theorem 2.2.8 and lemma 2.2.9 (b). From proposition 2.2.5 it follows that if $X = [1, \omega^\alpha \cdot n]$, then $X^{(\alpha)} = \{\omega^\alpha \cdot 1, \ldots, \omega^\alpha \cdot n\}$, which proves the second part. □

We finish this section with the remark that the second statement of corollary 2.2.10 is not true if we take $\alpha \geq \omega_1$. For example take $X = [1, \omega^{\omega_1} \cdot 2]$, i.e., $\alpha = \omega_1$. We first show that $\omega^{\omega_1} = \omega_1$. By proposition 2.1.4 (b), $\omega_1 \leq \omega^{\omega_1}$. Since ω_1 is a prime component (theorem 2.1.23) there is μ such that $\omega_1 = \omega^{\mu}$. If $\omega_1 < \omega^{\omega_1}$ then by proposition 2.1.4 (a), $\mu < \omega_1$, hence $\omega^{\mu} < \omega_1$ and we arrived at a contradiction.

Now put $A = X^{(\alpha)} = \{\omega_1, \omega_1 \cdot 2\}$, We prove that X/A and $[1, \omega^{\omega_1}] = [1, \omega_1]$ are not homeomorphic. To this end, notice that $(X/A) \setminus \{\infty\}$ contains two disjoint closed subsets E and F (namely $E = p([1, \omega_1))$ and $F = p([\omega_1 + 1, \omega_1 \cdot 2))$), such that the closures $\overline{E}$ and $\overline{F}$ in X/A have non-empty intersection. In $[1, \omega_1]$ for every pair E and F of disjoint closed subsets of $[1, \omega_1)$, the closures $\overline{E}$ and $\overline{F}$ in $[1, \omega_1]$ are disjoint ([24, Ex. 3.1.27]). Hence X/A and $[1, \omega_1]$ are not homeomorphic (see also [8, Ex. 1]).

§2.3. Factorizing function spaces

In this section we prove some results, which will be important tools later on. First we fix some notation and give some definitions. Let X be a space and $A \subset X$ closed. By $C_{p,A}(X)$ we denote the subspace of $C_p(X)$ of all functions vanishing on A. Whenever $A = \{a\}$ for some $a \in A$, $C_{p,\{a\}}(X)$ will be denoted by $C_{p,a}(X)$, so $C_{p,\infty}(X/A)$ is the subspace of $C_p(X/A)$ of all functions vanishing at ∞. For this kind of subspaces of $C_p^*(X)$ we use a similar notation. Furthermore, let $\{X_t : t \in T\}$ and $\{Y_s : s \in S\}$ be two families of spaces. For each $t \in T$ and $s \in S$ let E_t be a linear subspace of $C_p(X_t)$ and let F_s be a linear subspace of $C_p(Y_s)$ and let $k \in \mathbb{N}$. We call a linear function $\phi: \Pi_{t \in T} E_t \to \Pi_{s \in S} F_s$ a ***linear k-mapping*** if for all $(f_t)_{t \in T} \in \Pi_{t \in T} E_t$ with $f_t(X_t) \subset (-\frac{1}{k}, \frac{1}{k})$ for every $t \in T$ we have $(\pi_s \circ \phi)((f_t)_{t \in T})(Y_s) \subset (-1, 1)$ for every $s \in S$. We define ϕ to be a ***linear k-homeomorphism*** whenever ϕ is a homeomorphism such that both ϕ and ϕ^{-1} are linear k-mappings. Whenever there is a linear k-homeomorphism between $\Pi_{t \in T} E_t$ and $\Pi_{s \in S} F_s$ we write $\Pi_{t \in T} E_t \overset{k}{\sim} \Pi_{s \in S} F_s$. Notice that the composition of a linear k-homeomorphism and a linear l-homeomorphism is a linear kl-homeomorphism. The definition of linear k-mapping and linear k-homeomorphism can be found in [3] and can also be given for spaces of bounded continuous functions.

We now prove the following well-known theorem which will be used in the proof of proposition 2.3.2 and which will also be useful in chapter 4.

2.3.1 THEOREM (Dugundji [22]): *Let X be a metric space and A a closed subspace of X. Then there is a continuous linear function $\phi: C_p(A) \to C_p(X)$ such that for each $f \in C(A)$, $\phi(f) \mid A = f$ and $\phi(f)(X) \subset \operatorname{conv}(f(A))$.*

PROOF: First suppose $X \setminus A$ contains more than one point. Then for every $x \in X \setminus A$, there is a neighborhood V_x of x such that $\operatorname{diam} V_x < 1/2 d(x, A)$ and $V_x \neq X \setminus A$. Let $\mathcal{U}$ be a locally finite open refinement of the covering $\{V_x : x \in X \setminus A\}$ of $X \setminus A$. Notice that by construction $(X \setminus A) \setminus U \neq \emptyset$.

CLAIM: If $a \in A$ and V is a neighborhood of a, then there exists a neighborhood W of a such that if $U \cap W \neq \emptyset$ for some $U \in \mathcal{U}$, then $U \subset V$.

Let $\varepsilon = d(a, X \setminus V)$ and let $W = B(a, \varepsilon/2)$. Suppose that $U \cap W \neq \emptyset$ for some $U \in \mathcal{U}$, say $z \in U \cap W$. Choose $x \in X \setminus A$ such that $U \subset V_x$ and let $y \in U$. Now

$$d(x, a) \leq d(x, z) + d(z, a) < \frac{1}{2} d(x, A) + d(z, a) \leq \frac{1}{2} d(x, a) + d(z, a),$$

hence $d(x, a) < 2d(z, a)$. This implies

$$d(y, a) \leq d(y, z) + d(z, a) < \frac{1}{2} d(x, a) + d(z, a) < 2d(z, a) < \varepsilon,$$

hence $y \in V$. This proves the claim.

For each $U \in \mathcal{U}$, define $\lambda_U: X \setminus A \to \mathbb{R}$ by

$$\lambda_U(x) = \frac{d(x, (X \setminus A) \setminus U)}{\Sigma_{V \in \mathcal{U}} d(x, (X \setminus A) \setminus V)}.$$

First notice that $d(x, (X \setminus A) \setminus U)$ is defined for every $U \in \mathcal{U}$, because $(X \setminus A) \setminus U) \neq \emptyset$. Second for each $x \in X \setminus A$, there is a neighborhood W of x which intersects only finitely many elements of $\mathcal{U}$. Hence if we restrict λ_U to W, the sum in the denominator is finite, and since $\mathcal{U}$ covers $X \setminus A$, this sum is non-zero. We conclude that λ_U is a well-defined continuous function. Notice that $\Sigma_{U \in \mathcal{U}} \lambda_U \equiv 1$.

For each $U \in \mathcal{U}$, let $x_U \in U$ and $a_U \in A$ be such that $d(x_U, a_U) < 2d(x_U, A)$. Let $f \in C(A)$. Define $\tilde{f}: X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} \Sigma_{U \in \mathcal{U}} \lambda_U(x) f(a_U) & \text{if } x \in X \setminus A \\ f(x) & \text{if } x \in A \end{cases}.$$

By similar arguments as above it is easily seen that $\tilde{f}$ is well-defined and that $\tilde{f}|X\setminus A$ is continuous. So continuity of $\tilde{f}$ need only be verified at points of A. Let $x\in A$ and $\varepsilon>0$. There is $\delta>0$ such that for $y\in A$ with $d(x,y)<\delta$, we have $|f(x)-f(y)|<\varepsilon$. By the claim, there is a neighborhood $W\subset B(x,\delta)$ of x such that if $U\cap W\neq\emptyset$ for some $U\in\mathcal{U}$, then $U\subset B(x,\delta/3)$. We claim that for $y\in W$, $|\tilde{f}(x)-\tilde{f}(y)|<\varepsilon$. For $y\in A$, this is clear. So let $y\in W\cap X\setminus A$. Find $U_1,\ldots,U_n\in\mathcal{U}$ such that for $U\in\mathcal{U}$ we have $y\in U$ if and only if $U\in\{U_1,\ldots,U_n\}$. Then $\tilde{f}(y)=\Sigma_{i=1}^n\lambda_{U_i}(y)f(a_{U_i})$. For $i\leq n$, $U_i\cap W\neq\emptyset$, hence $d(x_{U_i},x)<\delta/3$. This implies

$$\begin{aligned} d(x,a_{U_i}) &\leq d(x,x_{U_i})+d(x_{U_i},a_{U_i})\\ &\leq d(x,x_{U_i})+2d(x_{U_i},A)\\ &\leq 3d(x,x_{U_i})<\delta. \end{aligned}$$

Hence $|f(x)-f(a_{U_i})|<\varepsilon$, so

$$\begin{aligned} |\tilde{f}(x)-\tilde{f}(y)| &= |f(x)-\Sigma_{i=1}^n\lambda_{U_i}(y)f(a_{U_i})|\\ &= |\Sigma_{i=1}^n\lambda_{U_i}(y)(f(x)-f(a_{U_i}))|\\ &\leq \Sigma_{i=1}^n\lambda_{U_i}(y)\,|f(x)-f(a_{U_i}))|\\ &< \Sigma_{i=1}^n\lambda_{U_i}(y)\cdot\varepsilon=\varepsilon. \end{aligned}$$

We conclude that $\tilde{f}$ is continuous. Obviously $\tilde{f}(X)\subset\operatorname{conv}(f(A))$ and $\tilde{f}|A=f$.

Define $\phi\colon C_p(A)\to C_p(X)$ by $\phi(f)=\tilde{f}$. By the above we have that ϕ is a well-defined function with the property that for each $f\in C(A)$, $\phi(f)|A=f$ and $\phi(f)(X)\subset\operatorname{conv}(f(A))$. The linearity of ϕ is a triviality. To prove that ϕ is continuous, it suffices to prove continuity at 0. Let $P\subset X$ be finite and $\varepsilon>0$. Let $Q=(P\cap A)\cup\{a_U: U\in\mathcal{U},\ U\cap P\neq\emptyset\}$. Then Q is a finite subset of A. It is easily seen that $\phi(<0,Q,\varepsilon>)\subset<0,P,\varepsilon>$.

Now assume $X\setminus A$ is empty or contains only one point. If it is empty, the theorem is obvious, so suppose $X\setminus A$ contains only one point x_0. Since A is closed, x_0 is isolated in X. Fix $x_1\in A$ and define $\phi\colon C_p(A)\to C_p(X)$ by $\phi(f)(x)=f(x)$ if $x\neq x_0$ and $\phi(f)(x_0)=f(x_1)$. Then ϕ is a well-defined linear function. In addition $\phi(f)|A=f$ and $\phi(f)(X)\subset\operatorname{conv}(f(A))$. We prove that ϕ is continuous. Take $P\subset X$ finite, $\varepsilon>0$ and $f\in C(A)$. If we let $Q=(P\cap A)\cup\{x_1\}$, then $\phi(<f,Q,\varepsilon>)\subset<\phi(f),P,\varepsilon>$. □

We now come to the important

2.3.2 PROPOSITION: *Let X be a metric space and let A be a closed subset of X. Then $C_p(X) \overset{2}{\sim} C_{p,A}(X) \times C_p(A)$.*

PROOF: Define $\rho\colon C_p(X) \to C_p(A)$ by $\rho(f) = f \mid A$. Notice that ρ is a continuous linear function. Because X is metric, by theorem 2.3.1 there is a continuous linear function $\xi\colon C_p(A) \to C_p(X)$ such that for each $f \in C_p(A)$, $\xi(f) \mid A = f$ and $\xi(f)(X) \subset \operatorname{conv}(f(A))$. Notice that $\rho \circ \xi = id_{C_p(A)}$.

Now define $\phi\colon C_p(X) \to C_{p,A}(X) \times C_p(A)$ by

$$\phi(f) = (f - (\xi \circ \rho)(f),\ \rho(f)).$$

We have to prove that ϕ is well-defined. Take an arbitrary $f \in C_p(X)$. It is obvious that $\rho(f) \in C_p(A)$ and that $f - (\xi \circ \rho)(f) \in C_p(X)$. Furthermore

$$(f - (\xi \circ \rho)(f)) \mid A = \rho(f - (\xi \circ \rho)(f)) = \rho(f) - (\rho \circ \xi \circ \rho)(f)) = \rho(f) - \rho(f) \equiv 0,$$

so $f - (\xi \circ \rho)(f) \in C_{p,A}(X)$.

That ϕ is continuous and linear is a triviality. We show that ϕ is a linear homeomorphism. For that define $\psi\colon C_{p,A}(X) \times C_p(A) \to C_p(X)$ by

$$\psi(f,\ g) = f + \xi(g).$$

It is trivial that ψ is well-defined, continuous and linear. Furthermore, as is easily seen, $\psi \circ \phi = id_{C_p(X)}$. We show that $\phi \circ \psi = id_{C_{p,A}(X) \times C_p(A)}$. Take $f \in C_{p,A}(X)$ and $g \in C_p(A)$. Notice that $\rho(f) = f \mid A \equiv 0$, hence by linearity of ξ, $(\xi \circ \rho)(f) = \xi(0) \equiv 0$. So

$$\begin{aligned}
(\phi \circ \psi)(f,\ g) &= \phi(f + \xi(g)) \\
&= (f + \xi(g) - (\xi \circ \rho)(f + \xi(g)),\ \rho(f + \xi(g))) \\
&= (f + \xi(g) - (\xi \circ \rho)(f) - (\xi \circ \rho \circ \xi)(g),\ \rho(f) + (\rho \circ \xi)(g)) \\
&= (f + \xi(g) - 0 - \xi(g),\ 0 + g) \\
&= (f,\ g),
\end{aligned}$$

i.e., ϕ is a linear homeomorphism.

The only thing left to prove is that ϕ and ψ are linear 2-mappings. We first prove it for ϕ. Let $f \in C_p(X)$ with $|f(x)| < 1/2$ for every $x \in X$. Then $\rho(f)(A) \subset (-1/2,\ 1/2)$ and hence $(\xi \circ \rho)(f)(X) \subset \operatorname{conv} \rho(f)(A) \subset (-1/2,\ 1/2)$. Let $x \in X$. Then

$$|\pi_1 \circ \phi(f)(x)| = |f(x) - (\xi \circ \rho)(f)(x)| \leq |f(x)| + |(\xi \circ \rho)(f)(x)| < 1,$$

and for $a \in A$,

$$|\pi_2 \circ \phi(f)(a)| = |\rho(f)(a)| = |f(a)| < 1/2,$$

so ϕ is a linear 2-mapping. Now take $f \in C_{p,A}(X)$ and $g \in C_p(A)$ such that $f(X) \subset (-1/2, 1/2)$ and $f(A) \subset (-1/2, 1/2)$. Then $\xi(g)(X) \subset \text{conv}(g(A)) \subset (-1/2, 1/2)$, so for $x \in X$,

$$|\psi(f, g)(x)| = |f(x) + \xi(g)(x)| \leq |f(x)| + |\xi(g)(x)| < 1/2 + 1/2 = 1,$$

hence ψ is a linear 2-mapping. This completes the proof of the proposition. □

2.3.3 LEMMA: *Let X be a space and let A be a closed subset of X. Then*

$$C_{p,A}(X) \overset{1}{\sim} C_{p,\infty}(X/A).$$

PROOF: For every function $f \in C_{p,A}(X)$ there is a unique function $\tilde{f} \in C_{p,\infty}(X/A)$ such that $\tilde{f} \circ p = f$ [24, p124]. If we now define ϕ: $C_{p,A}(X) \to C_{p,\infty}(X/A)$ by $\phi(f) = \tilde{f}$, then ϕ is a well-defined linear bijection. Since for $f \in C_{p,A}(X)$, $y_1, \ldots, y_n \in X/A$, $\varepsilon > 0$ and $x_i \in p^{-1}(y_i)$ $(i \leq n)$ it is easily seen that

$$\phi\langle f, \{x_1, \ldots, x_n\}, \varepsilon\rangle = \langle \phi(f), \{y_1, \ldots, y_n\}, \varepsilon\rangle,$$

it follows that ϕ is a linear homeomorphism. That ϕ is a linear 1-homeomorphism is a triviality. □

From the last lemma and proposition we have the useful

2.3.4 COROLLARY: *Let X be a metric space and let A be a closed subset of X. Then* $C_p(X) \overset{2}{\sim} C_{p,\infty}(X/A) \times C_p(A)$. □

The next three lemmas are used often in this monograph. The proofs are easy and left to the reader.

2.3.5 LEMMA: *If X and Y are homeomorphic spaces, then* $C_p(X) \overset{1}{\sim} C_p(Y)$. □

2.3.6 LEMMA: *If X and Y are spaces and A is a subspace of X, then* $C_{p,A}(X) \times C_p(Y) \overset{1}{\sim} C_{p,A}(X \oplus Y)$. □

Notice that all the given facts so far are also valid for spaces of bounded continuous functions. In lemma 2.3.7, this is only the case for the second statement as is shown in

section 4.6 (cf. example 4.6.6).

2.3.7 LEMMA: *If* $X=\oplus_{i=1}^{\infty}X_i$ *and* $Y=\oplus_{i=1}^{\infty}Y_i$ *such that for every* $i\in\mathbb{N}$, $C_p(X_i)\sim C_p(Y_i)$, *then* $C_p(X)\sim C_p(Y)$. *Moreover, if for every* $i\in\mathbb{N}$ $C_p(X_i)\overset{k}{\sim}C_p(Y_i)$, *then* $C_p(X)\overset{k}{\sim}C_p(Y)$. □

We now prove some properties of function spaces of ordinals. We use the following notation. For an ordinal α we denote by $C_{p,0}([1,\alpha])$ the subspace of $C_p([1,\alpha])$ of all continuous functions vanishing at α (i.e., $C_{p,0}([1,\alpha])=C_{p,\alpha}([1,\alpha])$).

2.3.8 LEMMA: *Let* $\alpha\geq 1$ *and* $\beta\geq 1$ *be ordinals. Then*

$$C_p([1,\alpha+\beta])\overset{1}{\sim}C_p([1,\alpha])\times C_p([1,\beta])\overset{1}{\sim}C_p([1,\beta])\times C_p([1,\alpha])\overset{1}{\sim}C_p([1,\beta+\alpha])$$

and

$$C_{p,0}([1,\alpha+\beta])\overset{1}{\sim}C_p([1,\alpha])\times C_{p,0}([1,\beta]).$$

PROOF: Since $[1,\alpha+\beta]\approx[1,\alpha]\oplus[1,\beta]$ (notice that $h:[1,\alpha+\beta]\to[1,\alpha]\oplus[1,\beta]$ defined by $h(\gamma)=\gamma$ if $\gamma\leq\alpha$ and $h(\gamma)=\gamma-\alpha$ if $\gamma>\alpha$, is a homeomorphism) we have by lemmas 2.3.5 and 2.3.6

$$C_p([1,\alpha+\beta])\overset{1}{\sim}C_p([1,\alpha]\oplus[1,\beta])\overset{1}{\sim}C_p([1,\alpha])\times C_p([1,\beta])$$

and

$$C_{p,0}([1,\alpha+\beta])\overset{1}{\sim}C_p([1,\alpha])\times C_{p,0}([1,\beta]).\ \square$$

2.3.9 LEMMA: *Let* $\alpha\geq\omega$ *be an ordinal. Then* $C_p([1,\alpha])\overset{2}{\sim}C_{p,0}([1,\alpha])$.

PROOF: Define $\phi:C_p([1,\alpha])\to C_{p,0}([1,\alpha])$ by $\phi(f)(\beta)=f(\beta-1)-f(\alpha)$ if $1<\beta\leq\alpha$ and $\phi(f)(1)=f(\alpha)$. Since $\beta-1=\beta$ for $\beta\geq\omega$, it easily follows that ϕ is well-defined. That ϕ is linear is a triviality. Now take $P\subset[1,\alpha]$ finite, $\varepsilon>0$ and $f\in C_p([1,\alpha])$. Let $Q=\{\beta-1:\beta\in P\setminus\{1\}\}\cup\{\alpha\}$. It is easily seen that $\phi(<f,Q,\varepsilon/2>)\subset<\phi(f),P,\varepsilon>$, hence ϕ is continuous.

Now define $\psi:C_{p,0}([1,\alpha])\to C_p([1,\alpha])$ by $\psi(f)(\beta)=f(1+\beta)+f(1)$. An easy verification shows that ψ is a well-defined continuous linear function.

We are done if we prove that $\psi=\phi^{-1}$. Let $f\in C_{p,0}([1,\alpha])$ and $\beta\in[1,\alpha]$. Notice that $1+(\beta-1)=\beta$, so if $\beta\neq 1$ then

$$\begin{aligned}(\phi\circ\psi)(f)(\beta)&=\psi(f)(\beta-1)-\psi(f)(\alpha)\\&=f(1+(\beta-1))+f(1)-f(1+\alpha)-f(1)\\&=f(\beta).\end{aligned}$$

Furthermore,

$$(\phi\circ\psi)(f)(1)=\psi(f)(\alpha)=f(1+\alpha)+f(1)=f(1),$$

which implies that $\phi\circ\xi=id_{C_{p,0}([1,\alpha])}$. Now let $f\in C_p([1,\alpha])$ and $\beta\in[1,\alpha]$. Notice that $(1+\beta)-1=\beta$, so

$$\begin{aligned}(\psi\circ\phi)(f)(\beta)&=\phi(f)(1+\beta)+\phi(f)(1)\\&=f((1+\beta)-1)-f(\alpha)+f(\alpha)\\&=f(\beta),\end{aligned}$$

which proves that $\psi\circ\phi=id_{C_p([1,\alpha])}$ and the lemma is proved. □

Notice that $[1,\alpha]$ is a metric space if $\alpha<\omega_1$, so in that case lemma 2.3.9 is an easy consequence of proposition 2.3.2 and lemma 2.3.8.

2.3.10 REMARK: All results stated in this section, are also valid for function spaces endowed with the compact-open topology, with the exception of lemma 2.3.3 and corollary 2.3.4. They are true for function spaces endowed with the compact-open topology under the additional assumption that A is compact.

§2.4. Separable metric zero-dimensional compact spaces

In [10] Bessaga and Pelczyński presented the following isomorphical classification of the spaces $C_0(X)$, for separable metric zero-dimensional compact spaces:

2.4.1 THEOREM (Bessaga and Pelczyński): *Let X and Y be separable metric zero-dimensional compact spaces. Then $C_0(X)\sim C_0(Y)$ if and only if one of the following holds:*

(*a*) *X and Y are finite and have the same number of elements.*

(*b*) *There are countable infinite ordinals α and β such that $X\approx[1,\alpha]$, $Y\approx[1,\beta]$ and* $\max(\alpha,\beta)<[\min(\alpha,\beta)]^{\omega}$.

(*c*) *X and Y are uncountable.*

Notice that for a compact space X, we always have that X is finite, or is uncountable or is homeomorphic to $[1, \alpha]$ for some countable infinite ordinal α (by theorem 2.2.8). Also, case (c) is a direct consequence of Miljutin's theorem ([47, page 379]). Bessaga and Pelczyński's proof of (c) is different, because they were not aware of Miljutin's result (see [47, page 380]).

In this section we prove that a similar classification can be derived if we replace $C_0(X)$ by $C_p(X)$. We first need to prove some properties of function spaces of ordinals.

2.4.2 LEMMA: *Let* $\omega \leq \alpha < \omega_1$ *be a prime component and* $n \in \mathbb{N}$. *Then* $C_p([1, \alpha \cdot n]) \sim C_p([1, \alpha])$.

PROOF: By theorem 2.1.21 there is an ordinal μ such that $\alpha = \omega^\mu$, so

$$\begin{aligned} C_p([1, \alpha \cdot n]) &\sim C_p(\{\alpha \cdot 1, \ldots, \alpha \cdot n\}) \times C_{p,0}([1, \alpha]) && \text{corollaries 2.2.10 and 2.3.4} \\ &\sim C_{p,0}([1, \alpha]) && \text{lemma 2.3.8} \\ &\sim C_p([1, \alpha]) && \text{lemma 2.3.9.} \square \end{aligned}$$

It is essential in this lemma that $\alpha < \omega_1$ (cf. the remark after corollary 2.2.10). In section 2.5 we will show that $C_p([1, \omega_1 \cdot 2])$ and $C_p([1, \omega_1])$ are not linearly homeomorphic (cf. theorem 2.5.13).

2.4.3 LEMMA: *Let* $\omega \leq \alpha < \omega_1$ *be an ordinal. Then* $C_p([1, \alpha]) \sim C_p([1, \alpha'])$.

PROOF: By corollary 2.1.20, $\alpha = \alpha' \cdot n + \gamma$ for some $n \in \mathbb{N}$ and $\gamma < \alpha'$. By theorem 2.1.15 $\gamma + \alpha' = \alpha'$, which implies that $\gamma + \alpha' \cdot n = \gamma + \alpha' + \alpha' \cdot (n-1) = \alpha' \cdot n$. So

$$\begin{aligned} C_p([1, \alpha]) &= C_p([1, \alpha' \cdot n + \gamma]) \\ &\sim C_p([1, \gamma + \alpha' \cdot n]) && \text{lemma 2.3.8 for } \gamma \neq 0 \\ &= C_p([1, \alpha' \cdot n]) \\ &\sim C_p([1, \alpha']) && \text{lemma 2.4.2.} \square \end{aligned}$$

We now come to the following result:

2.4.4 PROPOSITION: *Let* $\omega \leq \alpha < \omega_1$ *be an ordinal and let* $\alpha \leq \beta < \alpha^\omega$. *Then* $C_p([1, \alpha]) \sim C_p([1, \beta])$.

PROOF: By lemma 2.1.22 and lemma 2.4.3 we may assume that α and β are prime components. By theorem 2.1.21 there are ordinals μ and ν such that $\alpha=\omega^{\mu}$ and $\beta=\omega^{\nu}$. Since $\alpha\le\beta<\alpha^{\omega}$, by proposition 2.1.10a, $\mu\le\nu<\mu\cdot\omega$.

We prove the lemma by transfinite induction on ν. If $\nu=\mu$ it is a triviality, so let $\nu>\mu$ and suppose the lemma is true for every ordinal γ such that $\mu\le\gamma<\nu$.

Let $X=[1,\beta]=[1,\omega^{\nu}]$ and $A=X^{(\mu)}$. By proposition 2.2.5 $\kappa(X)=\nu+1$, so $\mu<\kappa(X)$. Hence by corollary 2.2.10 $X/A\approx[1,\omega^{\mu}]=[1,\alpha]$.

CLAIM: There are ordinals $1\le\gamma<\nu$ and $n\in\mathbb{N}$ such that $A\approx[1,\omega^{\gamma}\cdot n]$.

Indeed, since $\mu<\nu<\mu\cdot\omega$, there is $k\in\mathbb{N}\setminus\{1\}$ such that $\mu\cdot(k-1)<\nu\le\mu\cdot k$. So by proposition 2.2.5 and proposition 2.2.4 (a),

$$A^{(\mu\cdot(k-1))}=(X^{(\mu)})^{(\mu\cdot(k-1))}=X^{(\mu\cdot k)}\subset[1,\omega^{\mu\cdot k}]^{(\mu\cdot k)}=\{\omega^{\mu\cdot k}\},$$

and

$$A^{(1)}=X^{(\mu+1)}\supset[1,\omega^{\mu+1}]^{(\mu+1)}\neq\emptyset,$$

hence $2\le\kappa(A)\le\mu\cdot(k-1)+1$. Since $\kappa(A)$ is a successor, there is $1\le\gamma\le\mu\cdot(k-1)<\nu$ such that $\kappa(A)=\gamma+1$. So by theorem 2.2.8 there is $n\in\mathbb{N}$ such that $A\approx[1,\omega^{\gamma}\cdot n]$, which proves the claim.

By corollary 2.3.4, lemma 2.3.9 and the claim it follows that

$$C_p([1,\beta])\sim C_p([1,\omega^{\gamma}\cdot n])\times C_p([1,\alpha])$$
$$\sim C_p([1,\omega^{\gamma}])\times C_p([1,\alpha])\qquad\text{(since }\gamma\ge1\text{ and by lemma 2.4.3).}$$

If $\gamma<\mu$ then by lemma 2.3.8 and theorem 2.1.15

$$C_p([1,\beta])\sim C_p([1,\omega^{\gamma}+\alpha])=C_p([1,\alpha]).$$

If $\gamma\ge\mu$ then by the inductive hypothesis $C_p([1,\omega^{\gamma}])\sim C_p([1,\alpha])$, so by lemma 2.3.8 and lemma 2.4.2

$$C_p([1,\beta])\sim C_p([1,\alpha])\times C_p([1,\alpha])\sim C_p([1,\alpha\cdot2\sim C_p([1,\alpha]).\ \square$$

We can now easily derive the following:

2.4.5 COROLLARY: *Let $\omega\le\alpha\le\beta<\omega_1$ be ordinals. Then*

$C_p([1,\alpha])\sim C_p([1,\beta])$ *if and only if* $\beta<\alpha^{\omega}$.

(In particular if $\alpha=\omega^{\mu}$ and $\beta=\omega^{\nu}$ with $\mu\le\nu$, then $C_p([1,\alpha])\sim C_p([1,\beta])$ if and only if $\nu<\mu\cdot\omega$).

PROOF: If $\beta < \alpha^{\omega}$ then apply proposition 2.4.4. Suppose $C_p([1, \alpha]) \sim C_p([1, \beta])$. By corollary 1.2.21, it follows that $C_0([1, \alpha]) \sim C_0([1, \beta])$. By theorem 2.4.1 this implies $\beta < \alpha^{\omega}$. □

2.4.6 REMARK: If X is a separable metric compact space and A is a closed subset of X, then X/A is a separable metric compact space. This follows from the fact that the quotient map $p: X \to X/A$ is perfect because X is compact.

We are now able to prove the classification we mentioned at the beginning of this section.

2.4.7 THEOREM: *Let X and Y be separable metric zero-dimensional compact spaces. Then $C_p(X) \sim C_p(Y)$ if and only if one of the following holds:*

(*a*) *X and Y are finite and have the same number of elements.*

(*b*) *There are countable infinite ordinals α and β such that $X \approx [1, \alpha]$, $Y \approx [1, \beta]$ and $max(\alpha, \beta) < [min(\alpha, \beta)]^{\omega}$.*

(*c*) *X and Y are uncountable.*

PROOF: If $C_p(X) \sim C_p(Y)$ then by corollary 1.2.21 we have $C_0(X) \sim C_0(Y)$. So by theorem 2.4.1, (a), (b) or (c) holds.

Now suppose that (a), (b) or (c) holds.

Case 1: (a) holds.

Suppose X and Y both contain m points. Then $C_p(X) \sim \mathbb{R}^m \sim C_p(Y)$.

Case 2: (b) holds.

By corollary 2.4.5 we have the desired equivalence.

Case 3: (c) holds.

It is enough to prove that for every uncountable separable metric zero-dimensional compact space X we have $C_p(X) \sim C_p(C)$ where C is the Cantor discontinuum. Let X be such a space.

By the Cantor-Bendixson Theorem (theorem 2.2.6) and the fact that X is second countable, $X = D \cup S$ with D closed and dense in itself and S countable. Since X is uncountable, D is non-empty, so by the fact that C is the unique non-empty separable metric zero-dimensional compact space without isolated points ([15]), we have $D \approx C$. By the same characterization of C, we also have that $(X \times [1, \omega]) \times C \approx C$, so we can find a closed copy E of $X \times [1, \omega]$ in D. Now

$$\begin{aligned} C_p(X) &\sim C_{p,D}(X) \times C_p(D) && \text{by proposition 2.3.2} \\ &\sim C_{p,D}(X) \times C_p(D \oplus D) && \text{since } D \oplus D \approx C \oplus C \approx C \approx D \end{aligned}$$

$\sim C_{p,D}(X)\times C_p(D)\times C_p(D)$	by lemma 2.3.6
$\sim C_p(X)\times C_p(D)$	by proposition 2.3.2
$\sim C_p(X)\times C_{p,E}(D)\times C_p(E)$	by proposition 2.3.2
$\sim C_p(X\oplus E)\times C_{p,E}(D)$	by lemma 2.3.6
$\sim C_p(E)\times C_{p,E}(D)$	since $X\oplus E\approx E$
$\sim C_p(D)$	by proposition 2.3.2. □

2.4.8 REMARK: From theorem 2.4.1 and theorem 2.4.7 it follows that the classification is such that for any two separable metric compact zero-dimensional spaces X and Y it follows that $C_p(X)$ is linearly homeomorphic to $C_p(Y)$ if and only if $C_0(X)$ is linearly homeomorphic to $C_0(Y)$. In general this is not the case (see the remark after corollary 1.2.21 on page 29).

One of the steps in the proof Bessaga and Pelczyński gave of theorem 2.4.1 is proposition 2.4.4 for function spaces endowed with the compact-open topology (or the topology of uniform convergence) (cf. lemma 1 in [10]). Their proof of this result is quite different from ours. They used for example the fact that if a Banach space (and $C_0(X)$ is a Banach space if X is a compact ordinal) is the direct sum of two closed linear subspaces E and F, then it is isomorphic to $E\times F$. Recall that our spaces $C_p(X)$ are not Banach. Also, they did not use an inductive argument.

It is also possible to prove proposition 2.4.4 following the pattern of the proof of Bessaga and Pelczyński: They used the above property of Banach spaces to conclude, that if $\beta\leq\alpha<\omega_1$ then $C_0([1,\alpha\cdot\beta])\sim C_0([1,\alpha])$. However, for the topology of pointwise convergence we can prove this directly by the method of corollary 2.3.4, using the fact that $(\alpha\cdot\beta)'=\alpha'\cdot\beta'$. All the other statements that Bessaga and Pelczyński proved, are also valid for function spaces endowed with the topology of pointwise convergence. So then we are in a position from which we can derive proposition 2.4.4 with the same arguments as the ones of Bessaga and Pelczyński.

We now give some examples which were already announced in chapter 1.

2.4.9 EXAMPLE: We show as announced on page 21 that in general for spaces X and Y, and an effective linear function $\phi: C(X)\to C(Y)$ such that for each $y\in Y$, $\mathrm{supp}(y)\neq\emptyset$, $\mathrm{supp}: Y\to\mathcal{P}(X)\setminus\{\emptyset\}$ need not be USC.

Let $X=[1,\omega^2]$ and $Y=[1,\omega]$. By theorem 2.4.7 there is a linear homeomorphism

$\phi: C_p(X) \to C_p(Y)$. By corollary 1.2.15 (a), ϕ is effective and by proposition 1.4.3, for each $y \in Y$, $\text{supp}(y) \neq \emptyset$. We claim that $\text{supp}: Y \to \mathcal{P}(X) \setminus \{\emptyset\}$ is not USC.

Since $\text{supp}(\omega)$ is finite (lemma 1.4.1), there is an infinite clopen subset U of X which misses $\text{supp}(\omega)$. Let $V = X \setminus U$. Then $\text{supp}(\omega) \subset V$. If $\text{supp}: Y \to \mathcal{P}(X) \setminus \{\emptyset\}$ is USC, then there is $n \in \mathbb{N}$ such that $\text{supp}[n, \omega] \subset V$. We now have by proposition 1.4.3,

$$X = \text{supp}\, Y = \text{supp}\{1, ..., n\} \cup \text{supp}[n, \omega] \subset \text{supp}\{1, ..., n\} \cup V,$$

hence $U \subset \text{supp}\{1, ..., n\}$. Since U is infinite and $\text{supp}\{1, ..., n\}$ is finite, we have a contradiction. We conclude that $\text{supp}: Y \to \mathcal{P}(X) \setminus \{\emptyset\}$ is not USC.

2.4.10 EXAMPLE: In this example we show that the first countability condition in theorems 1.5.10 and 1.5.12 is essential.

Let $X = [1, \omega] \times \mathbb{N}$, $A = X^{(1)} = \{\omega\} \times \mathbb{N}$ and $Y = X/A$. Then X is clearly first countable and normal. Since the quotient map p between X and Y is closed (cf. page 60), Y is normal ([24, th 1.5.20]). The proof of the following claim is standard. For the sake of completeness it will be included.

CLAIM: Y is not first countable.

Indeed, suppose $\{U_n : n \in \mathbb{N}\}$ is a countable base at ∞ in Y. Let $n \in \mathbb{N}$. Then $p^{-1}(U_n)$ is open in X and $\{\omega\} \times \mathbb{N} \subset p^{-1}(U_n)$. So for every $i \in \mathbb{N}$ there is $\alpha_i^n < \omega$ such that $[\alpha_i^n, \omega] \times \{i\} \subset p^{-1}(U_n)$. Now let $U = \bigcup_{i=1}^{\infty} [\alpha_i^i + 1, \omega] \times \{i\}$. Since $A \subset U$, $\infty \in p(U)$ and $p^{-1}(p(U)) = U$, so $p(U)$ is a neighborhood of ∞ in Y. Hence there is $n \in \mathbb{N}$ such that $U_n \subset p(U)$, so $p^{-1}(U_n) \subset U$. Hence $\bigcup_{i=1}^{\infty} [\alpha_i^n, \omega] \times \{i\} \subset U$. But then $[\alpha_n^n, \omega] \subset [\alpha_n^n + 1, \omega]$, which is a contradiction.

Notice that for every space Z and for every $z \in Z$, $C_p(Z) \sim C_{p,z}(Z) \times \mathbb{R}$ (It is easily seen that the function $\phi: C_p(Z) \to C_{p,z}(Z) \times \mathbb{R}$ defined by $\phi(f) = f - f(z)$ is a linear homeomorphism), so $C_{p,\infty}(Y) \times \mathbb{R} \sim C_p(Y)$. Hence

$$\begin{aligned}
C_p(X) &\sim C_p([1, \omega] \oplus X) && \text{because } X \approx [1, \omega] \oplus X \\
&\sim C_p([1, \omega]) \times C_p(X) && \text{by lemma 2.3.6} \\
&\sim C_p([1, \omega]) \times \mathbb{R} \times C_p(X) && \text{by proposition 2.3.2 and lemma 2.3.9} \\
&\sim C_p(X) \times \mathbb{R} && \text{as above} \\
&\sim C_{p,A}(X) \times \mathbb{R} && \text{by lemma 2.3.7 and 2.3.9} \\
&\sim C_{p,\infty}(Y) \times \mathbb{R} && \text{by lemma 2.3.3} \\
&\sim C_p(Y). &&
\end{aligned}$$

For theorem 1.5.12 notice that $X^{(1)}$ is not countably compact, and $Y^{(1)}=\{\infty\}$ is countably compact. For theorem 1.5.10 notice that X and Y are paracompact (by [24, Th 5.1.3 and 5.1.33]), X is locally compact and Y is not locally compact.

Furthermore, notice that this example is a counterexample for the following statement (see also section 0.1): If X and Y are ℓ_p-equivalent spaces, then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$, where $\mathcal{P}$ is one of the properties: local compactness, first countability, second countability, metrizability, weight, or character.

2.4.11 EXAMPLE: In this example we show that theorem 1.5.12 is not true for the α-th derivative if α is not a prime component.

Let $\alpha<\omega_1$ be an ordinal which is not a prime component. Observe that in this situation $1\leq\alpha'<\alpha<\alpha'\cdot\omega$ (lemma 2.1.19). Hence $\omega^{\alpha'}<\omega^{\alpha}<(\omega^{\alpha'})^{\omega}$ and so by theorem 2.4.7 $C_p([1,\omega^{\alpha'}])\sim C_p([1,\omega^{\alpha}])$. So if we now let $X=\oplus_{i=1}^{\infty}[1,\omega^{\alpha'}]_i$ and $Y=\oplus_{i=1}^{\infty}[1,\omega^{\alpha}]_i$, then $C_p(X)\sim C_p(Y)$ (lemma 2.3.7). In addition, X and Y are normal and first countable, but $Y^{(\alpha)}\approx\mathbb{N}$ (this follows easy from proposition 2.2.5) which is not countably compact, and $X^{(\alpha)}=\emptyset$ which is countably compact.

This observation leads us to the following

Question: Let X and Y be ℓ_p-equivalent spaces which are both normal and first countable. Let $\alpha\geq\omega$ be a prime component. Is it true that $X^{(\alpha)}$ is countably compact if and only if $Y^{(\alpha)}$ is countably compact?

Finally we remark that the first part of this section (until theorem 2.4.7) is taken from [3]. The examples 2.4.10 and 2.4.11 can be found in [5].

§2.5. Compact ordinals

In this section we present an isomorphical classification of the function spaces $C_p(X)$, where $X=[1,\alpha]$ for some ordinal α. We call such spaces ***compact ordinal spaces***. It turns out that this classification is similar to the one Kislyakov gave for the spaces $C_0(X)$ (with X a compact ordinal space) in [34]. Our proof is similar to his, only some modifications are necessary.

It turns out that Kislyakov made a mistake in his proof. In this section we will identify this mistake and correct it.

Let us first present the classification of Kislyakov. For that we need some

definitions. Let X and Y be Banach spaces. We say that X and Y have ***the same linear dimension***, if each of them is isomorphic to a subspace of the other. X ***has smaller linear dimension than*** Y if X is isomorphic to a subspace of Y, but Y is not isomorphic to any subspace of X. Notice that isomorphic spaces have the same linear dimension.

2.5.1 THEOREM: *Let α and β be ordinals.*
If α and β have different power, then

(*a*) $C_0([1, \alpha])$ *and* $C_0([1, \beta])$ *do not have the same linear dimension and so they are not linearly homeomorphic.*

If α and β have the same power and ξ is the initial ordinal of that power, then

(*b*) ([34]) *If* $\xi = \omega$, *or* ξ *is a singular ordinal or both* $\alpha, \beta \geq \xi^2$, *then* $C_0([1, \alpha]) \sim C_0([1, \beta])$ *if and only if* $\max(\alpha, \beta) < [\min(\alpha, \beta)]^\omega$ *if and only if* $C_0([1, \alpha])$ *and* $C_0([1, \beta])$ *have the same linear dimension.*

(*c*) ([34]) *If ξ is an uncountable regular ordinal and $\alpha, \beta \in [\xi, \xi^2]$, fix ordinals $\alpha_1, \beta_1 \leq \xi$ and $\gamma, \delta < \xi$ such that $\alpha = \xi \cdot \alpha_1 + \gamma$ and $\beta = \xi \cdot \beta_1 + \delta$. Then $C_0([1, \alpha]) \sim C_0([1, \beta])$ if and only if $\overline{\alpha_1} = \overline{\beta_1}$ if and only $C_0([1, \alpha])$ and $C_0([1, \beta])$ have the same linear dimension.*

(*d*) *If ξ is an uncountable regular ordinal, $\alpha < \xi^2$ and $\beta \geq \xi^2$, then $C_0([1, \alpha])$ and $C_0([1, \beta])$ are not linearly homeomorphic.*

Notice that the case $\xi = \omega$ in theorem 2.5.1 (b) is just Bessaga and Pelczyński's result stated in theorem 2.4.1. Furthermore theorem 2.5.1 (c) was proved by Semadeni in [46] for ordinals α and β satisfying $\omega_1 \leq \alpha, \beta \leq \omega_1 \cdot \omega$.

In fact Kislyakov only stated theorem 2.5.1 (b) and (c), so we will now prove part (a) and part (d). Before being able to prove this, we need to formulate the following lemma proved by Bessaga and Pelczyński in [10].

2.5.2 LEMMA: *Let α be an ordinal. If for every $\gamma < \alpha$, $C_0([1, \gamma])$ has smaller linear dimension than $C_0([1, \alpha])$, then $C_0([1, \alpha])$ has smaller linear dimension than $C_0([1, \alpha^\omega])$.*

The following corollary to lemma 2.5.2 is also useful in section 2.6, and is stated without proof by Bessaga and Pelczyński in [10] and by Kislyakov in [34]. For the sake of completeness, we will present its proof.

2.5.3 COROLLARY ([34, lemma 1.3]): *Let α and β be ordinals. If $\beta \geq \alpha^\omega$, then $C_0([1, \alpha])$ has smaller linear dimension than $C_0([1, \beta])$.*

PROOF: Let α_1 be the smallest ordinal such that $C_0([1, \alpha_1])$ and $C_0([1, \alpha])$ have the same linear dimension. Then for every $\gamma < \alpha_1$, $C_0([1, \gamma])$ has smaller linear dimension than $C_0([1, \alpha_1])$. By lemma 2.5.2 it follows that $C_0([1, \alpha_1])$ has smaller linear dimension than $C_0([1, \alpha_1^\omega])$. Since $\alpha_1 \leq \alpha$, it follows that $\alpha_1^\omega \leq \alpha^\omega \leq \beta$. But then it easily follows that $C_0([1, \alpha])$ has smaller linear dimension than $C_0([1, \beta])$. □

We are now able to prove theorem 2.5.1 (a) and (d).

PROOF of theorem 2.5.1 (a): Without loss of generality we may assume that $\bar{\alpha} < \bar{\beta}$. First suppose $\alpha \geq \omega$. Since $\overline{\alpha^\omega} = \bar{\alpha}$ (proposition 2.1.13), it follows that $\alpha^\omega < \beta$. So by corollary 2.5.3 it follows that $C_0([1, \alpha])$ has smaller linear dimension than $C_0([1, \beta])$, which implies that $C_0([1, \alpha])$ is not linearly homeomorphic to $C_0([1, \beta])$.

Now suppose that α is finite. As is easily seen, the algebraic dimension of $C_0([1, \alpha])$ is finite and smaller than the algebraic dimension of $C_0([1, \beta])$, hence $C_0([1, \alpha])$ and $C_0([1, \beta])$ are not linearly homeomorphic. □

PROOF of theorem 2.5.1 (d): First suppose that $\beta < \xi^\omega$. Since $\xi^2 \leq \beta < \xi^\omega = (\xi^2)^\omega$, by theorem 2.5.1, $C_0([1, \beta]) \sim C_0([1, \xi^2])$. Let $\alpha = \xi \cdot \alpha_1 + \gamma$ with $\gamma < \xi$ (proposition 2.1.2). Notice that $\alpha_1 < \xi$, so $\overline{\alpha_1} < \bar{\xi}$, because ξ is initial. From theorem 2.5.1 (c) it now follows that $C_0([1, \alpha])$ and $C_0([1, \xi^2])$ are not linearly homeomorphic.

Now suppose $\beta \geq \xi^\omega$. Since $\alpha < \xi^2$, $\alpha^\omega \leq (\xi^2)^\omega = \xi^\omega \leq \beta$. So by corollary 2.5.3, $C_0([1, \alpha])$ has smaller linear dimension than $C_0([1, \beta])$, which implies that $C_0([1, \alpha])$ and $C_0([1, \beta])$ are not linearly homeomorphic. □

Now we are going to prove that the same classification holds for the spaces $C_p([1, \alpha])$. For that we first have to give some definitions.

For a compact space X and $f \in C_p(X)$, let $\|f\| = \sup_{x \in X} |f(x)|$. Let $\{X_t : t \in T\}$ be a family of compact spaces and for each $t \in T$, let E_t be a linear subspace of $C_p(X_t)$. By $\Pi^*_{t \in T} E_t$ we denote the linear subspace of $\Pi_{t \in T} E_t$ consisting of all points $f = (f_t)_{t \in T}$ such that for each $\varepsilon > 0$, the set $\{t \in T : \|f_t\| \geq \varepsilon\}$ is finite. If $E_t = E$ for every $t \in T$, we write $\Pi^*_{\mathfrak{m}} E$ instead of $\Pi^*_{t \in T} E_t$, where $\mathfrak{m} = |T|$. Notice that if T is finite, then $\Pi_{t \in T} E_t = \Pi^*_{t \in T} E_t$.

The notion of linear k-mapping (cf. section 2.3) can also be defined for the spaces $\Pi^*_{t \in T} E_t$. We then have the following

2.5.4 LEMMA: *For each $i \in I$ let $\{X_{t,i} : t \in T\}$ and $\{Y_{s,i} : s \in S\}$ be two families of compact spaces and let for each $i \in I$, $t \in T$ and $s \in S$, $E_{t,i}$ and $F_{s,i}$ be linear subspaces of $C_p(X_{t,i})$ and $C_p(Y_{s,i})$, respectively. Suppose that for each $i \in I$, $\Pi^*_{t \in T} E_{t,i} \overset{k}{\sim} \Pi^*_{s \in S} F_{s,i}$. Then $\Pi^*_{(t,i) \in T \times I} E_{t,i} \overset{k}{\sim} \Pi^*_{(s,i) \in S \times I} F_{s,i}$.*

PROOF: For each $i \in I$ let $\phi_i : \Pi^*_{t \in T} E_{t,i} \to \Pi^*_{s \in S} F_{s,i}$ be a linear k-homeomorphism. Define $\phi : \Pi^*_{(t,i) \in T \times I} E_{t,i} \to \Pi^*_{(s,i) \in S \times I} F_{s,i}$ by

$$\phi((f_{(t,i)})_{(t,i) \in T \times I}) = ((\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T}))_{(s,i) \in S \times I}.$$

We prove that ϕ is a linear k-homeomorphism.

CLAIM 1: ϕ is well-defined.

Indeed, let $\varepsilon > 0$ and $(f_{(t,i)})_{(t,i) \in T \times I} \in \Pi^*_{(t,i) \in T \times I} E_{t,i}$. It is a triviality that $(\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T}) \in F_{(s,i)}$. Notice that $J = \{i \in I : (\exists\, t \in T)(\|f_{(t,i)}\| \geq \varepsilon/k)\}$ is finite. Let $i \in I \setminus J$. Then for every $t \in T$, $f_{(t,i)}(X_{t,i}) \subset (-\varepsilon/k, \varepsilon/k)$. Since ϕ_i is a linear k-homeomorphism, for every $s \in S$

$$(\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T})(Y_{s,i}) \subset (-\varepsilon, \varepsilon).$$

So $\{i \in I : (\exists\, s \in S)(\|(\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T})\| \geq \varepsilon)\}$ is finite. Since

$$\phi_i((f_{(t,i)})_{t \in T}) \in \Pi^*_{s \in S} F_{s,i}$$

for every $i \in I$, $\{s \in S : \|(\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T})\| \geq \varepsilon\}$ is finite. But this implies that

$$\{(s,i) \in S \times I : \|(\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T})\| \geq \varepsilon\}$$

is finite as well, which proves the claim.

Since π_s and ϕ_i are linear and continuous, it is clear that ϕ is linear and continuous.

CLAIM 2: ϕ is a linear k-mapping.

Indeed, let $(f_{(t,i)})_{(t,i) \in T \times I} \in \Pi^*_{(t,i) \in T \times I} E_{t,i}$ be such that $f_{(t,i)}(X_{t,i}) \subset (-1/k, 1/k)$ for each pair $(t,i) \in T \times I$. Then

$$(\pi_{(s,i)} \circ \phi)((f_{(t,i)})_{(t,i) \in T \times I})(Y_{t,i}) = (\pi_s \circ \phi_i)((f_{(t,i)})_{t \in T})(Y_{t,i}) \subset (-1, 1)$$

for each pair $(s,i) \in S \times I$, because ϕ_i is a linear k-homeomorphism. This proves claim 2.

Now define $\psi : \Pi^*_{(s,i) \in S \times I} F_{s,i} \to \Pi^*_{(t,i) \in T \times I} E_{t,i}$ by

$$\psi((g_{(s,i)})_{(s,i)\in S\times I}) = ((\pi_t \circ \phi_i^{-1})((g_{(s,i)})_{s\in S}))_{(t,i)\in T\times I}.$$

The proof that ψ is a well-defined continuous linear k-mapping is exactly the same as the proof for ϕ. Furthermore it is easily seen that $\psi = \phi^{-1}$, so ϕ is a linear k-homeomorphism. □

In the sequel we denote $\Pi^*_{(t,i)\in T\times I} E_{t,i}$ also by $\Pi^*_{t\in T}\Pi^*_{i\in I} E_{t,i}$ or by $\Pi^*_{i\in I}\Pi^*_{t\in T} E_{t,i}$. Notice that this is not the same as $\Pi^*_{t\in T}(\Pi^*_{i\in I} E_{t,i})$, because the latter product is not defined. Lemma 2.5.4 now gives us the following: If for each $i \in I$,

$$\Pi^*_{t\in T} E_{t,i} \overset{k}{\sim} \Pi^*_{s\in S} F_{s,i},$$

then

$$\Pi^*_{i\in I}\Pi^*_{t\in T} E_{t,i} \overset{k}{\sim} \Pi^*_{i\in I}\Pi^*_{s\in S} E_{s,i}.$$

2.5.5 LEMMA: *Let $\{X_t : t \in T\}$ be a family of compact spaces and let for each $t \in T$, E_t be a linear subspace of $C_p(X_t)$. Let $S \subset T$. Then $\Pi^*_{t\in T} E_t \overset{1}{\sim} \Pi^*_{t\in S} E_t \times \Pi^*_{t\in T\setminus S} E_t$.*

PROOF: Define $\phi: \Pi^*_{t\in T} E_t \to \Pi^*_{s\in S} E_s \times \Pi^*_{t\in T\setminus S} E_t$ by $\phi((f_t)_{t\in T}) = ((f_t)_{t\in S}, (f_t)_{t\in T\setminus S})$. It is a triviality that ϕ is a well-defined continuous linear 1-mapping as well. The inverse ϕ^{-1} of ϕ can also be defined canonically and is a continuous linear 1-mapping as well. So ϕ is a linear 1-homeomorphism. □

The next lemma is the main tool in this section.

2.5.6 LEMMA: *Let γ be a limit ordinal. Let $(\lambda_\xi)_{0\le\xi\le\gamma}$ be a strictly increasing sequence such that*

1) $\lambda_{\xi+1} - \lambda_\xi \ge \omega$ $(\xi \in [0, \gamma))$,
2) $\lambda_\xi = \lim_{\eta<\xi} \lambda_\eta$ $(\xi \in (0, \gamma]$ *a limit ordinal)*,
3) $\lambda_0 = 0$.

Then $C_{p,0}([1, \lambda_\gamma]) \overset{2}{\sim} C_{p,0}([1, \gamma]) \times \Pi^*_{\xi<\gamma} C_{p,0}([1, \lambda_{\xi+1} - \lambda_\xi])$.

PROOF: Let

$$X = \{f \in C_{p,0}([1, \lambda_\gamma]) : f \text{ is constant on each interval } (\lambda_\xi, \lambda_{\xi+1}]\ (\xi \in [0, \gamma))\}$$

and

$$Y = \{f \in C_{p,0}([1, \lambda_\gamma]) : f(\lambda_\xi) = 0\ (\xi \in (0, \gamma])\}.$$

CLAIM 1: $C_{p,0}([1, \lambda_\gamma]) \overset{2}{\sim} X \times Y$.

Indeed, define $\phi_1: C_{p,0}([1, \lambda_\gamma]) \to X$ by

$$\phi_1(f) \restriction (\lambda_\xi, \lambda_{\xi+1}] = f(\lambda_{\xi+1}) \ (\xi \in [0, \gamma)).$$

and

$$\phi_1(f)(\lambda_\xi) = f(\lambda_\xi) \ (\xi \in (0, \gamma] \text{ a limit ordinal}).$$

To prove that ϕ_1 is well-defined, we need to show that $\phi_1(f)$ is continuous at λ_ξ with $\xi \in (0, \gamma]$ a limit ordinal. So let $\varepsilon > 0$. Since $\lambda_\xi = \lim_{\eta<\xi} \lambda_\eta$ and since f is continuous at λ_ξ, there is $\eta < \xi$ such that $f((\lambda_\eta, \lambda_\xi]) \subset (-\varepsilon + f(\lambda_\xi), \varepsilon + f(\lambda_\xi))$. But then also

$$\phi_1(f)((\lambda_\eta, \lambda_\xi]) \subset (-\varepsilon + f(\lambda_\xi), \varepsilon + f(\lambda_\xi)).$$

That ϕ_1 is linear is a triviality. We prove that ϕ_1 is continuous. Let $P = \{\alpha_i : i \leq n\} \subset [1, \lambda_\gamma]$ be finite, $\varepsilon > 0$, and $f \in C_{p,0}([1, \lambda_\gamma])$. For $i \leq n$, if $\alpha_i = \lambda_\xi$ for some ξ, put $\beta_i = \alpha_i$, otherwise let λ_ξ be such that $\alpha_i \in (\lambda_\xi, \lambda_{\xi+1}]$ and put $\beta_i = \lambda_{\xi+1}$. Let $Q = \{\beta_i : i \leq n\}$. Then a simple calculation shows that $\phi_1(\langle f, Q, \varepsilon\rangle) \subset \langle \phi_1(f), P, \varepsilon\rangle$, which proves that ϕ_1 is continuous.

Now define $\phi_2: C_{p,0}([1, \lambda_\gamma]) \to Y$ by $\phi_2(f) = f - \phi_1(f)$. It is easily seen that ϕ_2 is well-defined, continuous and linear. It follows that the map $\phi: C_{p,0}([1, \lambda_\gamma]) \to X \times Y$ defined by $\phi(f) = (\phi_1(f), \phi_2(f))$ is also well-defined, linear and continuous .

We prove that ϕ is a linear 2-mapping. For that let $f \in C_{p,0}([1, \lambda_\gamma])$ with $f([1, \lambda_\gamma]) \subset (-1/2, 1/2)$. Let $\alpha \in [1, \lambda_\gamma]$. Then $|\phi_1(f)(\alpha)| < 1/2$ and therefore

$$|\phi_2(f)(\alpha)| = |f(\alpha) - \phi_1(f)(\alpha)| \leq |f(\alpha)| + |\phi_1(f)(\alpha)| < 1/2 + 1/2 = 1,$$

which proves that ϕ is a linear 2-mapping.

Now define $\psi: X \times Y \to C_{p,0}([1, \lambda_\gamma])$ by $\psi(f, g) = f + g$. It is evident that ψ is a well-defined continuous linear 2-mapping. Furthermore one can simply derive that $\psi = \phi^{-1}$, and so we conclude that ϕ is a linear 2-homeomorphism, which proves claim 1.

CLAIM 2: $X \overset{l}{\sim} C_{p,0}([1, \gamma])$.

Indeed, define $\phi: X \to C_{p,0}([1, \gamma])$ by $\phi(f)(\xi) = f(\lambda_\xi)$. Since by (2) the function $\xi \to \lambda_\xi$ is continuous, ϕ is well-defined. It is easily seen that ϕ is a continuous, linear 1-mapping. Define $\psi: C_{p,0}([1, \gamma]) \to X$ by $\psi(f) \restriction (\lambda_\xi, \lambda_{\xi+1}] = f(\xi + 1)$ for $\xi \in [0, \gamma)$ and $\psi(f)(\lambda_\xi) = f(\xi)$ for $\xi \in (0, \gamma]$ a limit ordinal. It is a triviality that ψ is a well-defined, continuous, linear 1-mapping, which is the inverse of ϕ. Whence ϕ is a linear 1-homeomorphism.

CLAIM 3: $Y \overset{l}{\sim} \Pi^*_{\xi<\gamma} C_{p,0}([1, \lambda_{\xi+1} - \lambda_\xi])$.

Indeed, define $\phi: Y \to \Pi^*_{\xi<\gamma} C_{p,0}((\lambda_\xi, \lambda_{\xi+1}])$ by $(\pi_\xi \circ \phi)(f) = f \restriction (\lambda_\xi, \lambda_{\xi+1}]$ $(\xi < \gamma)$.

To see that ϕ is well-defined, we assume there are $f \in Y$, $\varepsilon > 0$ and $\xi_1 < \xi_2 < \cdots < \gamma$ such that $\|(\pi_{\xi_n} \circ \phi)(f)\| \geq \varepsilon$ for each $n \in \mathbb{N}$. Let $\xi = \lim_{n \in \mathbb{N}} \xi_n$. Then $\xi \leq \gamma$. For each $n \in \mathbb{N}$ find $\alpha_n \in (\lambda_{\xi_n}, \lambda_{\xi_n+1}]$ with $|f(\alpha_n)| \geq \varepsilon$. Then $|f(\lambda_\xi)| \geq \varepsilon$, which is a contradiction, because $f \in Y$. It is easily seen that ϕ is a continuous linear 1-mapping.

Define ψ: $\Pi^*_{\xi<\gamma} C_{p,0}((\lambda_\xi, \lambda_{\xi+1}]) \to Y$ by $\psi((f_\xi)_{\xi<\gamma})(\beta) = f_\xi(\beta)$ if $\beta \in (\lambda_\xi, \lambda_{\xi+1}]$ and $\psi((f_\xi)_{\xi<\gamma})(\lambda_\xi) = 0$ if $\xi \in (0, \gamma]$ is a limit ordinal. It is evident that ψ is a well-defined continuous linear 1-mapping, which is the inverse of ϕ. Thus ϕ is a linear 1-homeomorphism. The claim is now proved since $(\lambda_\xi, \lambda_{\xi+1}] \approx [1, \lambda_{\xi+1} - \lambda_\xi]$.

It is clear that the claims 1, 2, and 3 establish the proof of lemma 2.5.6. □

2.5.7 LEMMA: *Let μ be an infinite ordinal and γ a limit ordinal such that $\gamma = \mu$ or $\gamma + \mu = \mu$. Then $C_{p,0}([1, \mu \cdot \gamma]) \overset{4}{\sim} \Pi^*_\gamma C_{p,0}([1, \mu])$.*

PROOF: By lemma 2.5.6, applied to the sequence $\lambda_\xi = \mu \cdot \xi$ for $\xi \leq \gamma$, we have

$$\begin{aligned} C_{p,0}([1, \mu\cdot\gamma]) &\overset{2}{\sim} C_{p,0}([1, \gamma]) \times \Pi^*_{\xi<\gamma} C_{p,0}([1, \mu(\xi+1) - \mu\cdot\xi]) \\ &= C_{p,0}([1, \gamma]) \times \Pi^*_\gamma C_{p,0}([1, \mu]). \end{aligned}$$

Now suppose $\gamma = \mu$. Then by lemma 2.5.5 $C_{p,0}([1, \mu\cdot\gamma]) \overset{2}{\sim} \Pi^*_\gamma C_{p,0}([1, \mu])$. If $\gamma + \mu = \mu$, then

$$\begin{aligned} C_{p,0}([1, \mu\cdot\gamma]) &\overset{2}{\sim} C_{p,0}([1, \gamma]) \times C_{p,0}([1, \mu]) \times \Pi^*_\gamma C_{p,0}([1, \mu]) && \text{(lemma 2.5.5)} \\ &\overset{2}{\sim} C_{p,0}([1, \mu]) \times \Pi^*_\gamma C_{p,0}([1, \mu]) && \text{(lemma 2.3.8, 2.3.9)} \\ &\overset{1}{\sim} \Pi^*_\gamma C_{p,0}([1, \mu]) && \text{(lemma 2.5.5).} \ \square \end{aligned}$$

2.5.8 LEMMA: *Let α be an initial ordinal and γ a limit ordinal with $\overline{\gamma} \leq \overline{\alpha}$. Then there exists a subset M of $[2, \gamma)$ consisting of successors such that*

$$C_{p,0}([1, \alpha^\gamma]) \overset{4}{\sim} \Pi^*_{\mu \in M} C_{p,0}([1, \alpha^\mu]).$$

PROOF: Let $\beta = cf(\gamma)$. Since β is initial, $cf(\gamma) \leq \gamma$, and $\overline{\gamma} \leq \overline{\alpha}$, we have $\beta \leq \alpha$.

CLAIM: There is a strictly increasing sequence $\{\mu_\xi : \xi \leq \beta\}$ in $[2, \gamma]$ such that μ_ξ is a successor for each successor $\xi \leq \beta$, $\mu_\xi = \lim_{\eta<\xi} \mu_\eta$ for a limit ordinal $\xi \leq \beta$, and $\mu_\beta = \gamma$,

Indeed, let ϕ be an increasing β-sequence such that $\lim_{\xi<\beta} \phi(\xi) = \gamma$ and $\phi(1) \geq 1$. Let $\xi \leq \beta$ and take $\mu_\xi = \phi(\xi) + 1$ if ξ is a limit ordinal, otherwise take $\mu_\xi = \lim_{\eta<\xi} \phi(\xi)$. It is a triviality that $\{\mu_\xi : \xi \leq \beta\}$ is as required.

Since α is a prime component (theorem 2.1.23), $\alpha^{\mu_{\xi+1}} - \alpha^{\mu_\xi} = \alpha^{\mu_{\xi+1}}$ and $\beta + \alpha^{\mu_1} = \alpha^{\mu_1}$ (by theorem 2.1.15 and the facts that $\mu_1 \geq 2$ and $\beta \leq \alpha$). By applying lemma 2.5.6 to the sequence $\lambda_\xi = \alpha^{\mu_\xi}$ for $\xi \in (0, \beta]$ and $\lambda_0 = 0$, we get

$$\begin{aligned} C_{p,0}([1, \alpha^\gamma]) &\overset{2}{\sim} C_{p,0}([1, \beta]) \times \Pi^*_{\xi<\beta} C_{p,0}([1, \alpha^{\mu_{\xi+1}} - \alpha^{\mu_\xi}]) \\ &= C_{p,0}([1, \beta]) \times \Pi^*_{\xi<\beta} C_{p,0}([1, \alpha^{\mu_{\xi+1}}]) \\ &\overset{2}{\sim} C_{p,0}([1, \beta + \alpha^{\mu_1}]) \times \Pi^*_{1\leq\xi<\beta} C_{p,0}([1, \alpha^{\mu_{\xi+1}}]) \\ &= C_{p,0}([1, \alpha^{\mu_1}]) \times \Pi^*_{1\leq\xi<\beta} C_{p,0}([1, \alpha^{\mu_{\xi+1}}]) \\ &\overset{1}{\sim} \Pi^*_{\xi<\beta} C_{p,0}([1, \alpha^{\mu_{\xi+1}}]). \end{aligned}$$

We applied lemma 2.5.6 for the first equivalence, lemmas 2.5.5, 2.3.9 and 2.3.8 for the third one and lemma 2.5.5 for the last one. Now take $M = \{\mu_{\xi+1} : \xi < \beta\}$. □

2.5.9 LEMMA: *Let ω_γ be a singular ordinal. Then there exist $\beta < \omega_\gamma$ and a strictly increasing β-sequence ϕ such that*

(*a*) $\lim_{\xi<\beta}\phi(\xi) = \gamma$,

(*b*) $\omega_{\phi(\xi)} \geq \beta$ *for every* $\xi < \beta$,

(*c*) $C_{p,0}([1, \omega_\gamma]) \overset{4}{\sim} \Pi^*_{\xi<\beta} C_{p,0}([1, \omega_{\phi(\xi+1)}])$.

PROOF: Since ω_γ is singular, γ is a limit ordinal (corollary 2.1.11). Furthermore $\gamma \leq \omega_\gamma$.

CLAIM: There is an ordinal β and a strictly increasing β-sequence ϕ such that $\beta < \omega_\gamma$, $\lim_{\xi<\beta}\phi(\xi) = \gamma$, $\omega_{\phi(\xi)} \geq \beta$ for every $\xi < \beta$, and $\lim_{\xi<\eta}\phi(\xi) = \phi(\eta)$ if η is a limit ordinal.

For the proof of the claim we consider two cases.

Case 1: $\gamma < \omega_\gamma$.

Let $\xi_0 = \min\{\xi : \gamma < \omega_\xi < \omega_\gamma\}$ and notice that $\xi_0 < \gamma$. Put $\beta = \gamma - \xi_0$ and $\phi(\xi) = \xi_0 + \xi$. Then $\beta \leq \gamma < \omega_\gamma$, $\lim_{\xi<\beta}\phi(\xi) = \xi_0 + \beta = \gamma$, and by the definition of ϕ, $\omega_{\phi(\xi)} \geq \omega_{\xi_0} > \gamma \geq \beta$. That $\lim_{\xi<\eta}\phi(\xi) = \phi(\eta)$ if η is a limit ordinal is a triviality.

Case 2: $\gamma = \omega_\gamma$.

Let $\beta = cf(\omega_\gamma)$. Since ω_γ is singular, $\beta < \omega_\gamma$. Let ϕ_1 be a strictly increasing β-sequence such that $\lim_{\xi<\beta}\phi_1(\xi) = \omega_\gamma = \gamma$. As in the proof of lemma 2.5.8, we may assume that $\lim_{\xi<\eta}\phi(\xi) = \phi(\eta)$ if η is a limit ordinal. Now put $\phi(\xi) = \beta + \phi_1(\xi)$ for $\xi < \beta$. Then $\lim_{\xi<\beta}\phi(\xi) = \beta + \omega_\gamma = \omega_\gamma$ (theorem 2.1.15 and 2.1.23) and $\omega_{\phi(\xi)} \geq \omega_\beta \geq \beta$. This proves the claim.

For the following notice that $\omega_{\phi(\xi+1)} - \omega_{\phi(\xi)} = \omega_{\phi(\xi+1)}$ (theorems 2.1.15 and 2.1.23) and that $\beta + \omega_{\phi(1)} = \omega_{\phi(1)}$ (because $\beta \leq \omega_{\phi(0)} < \omega_{\phi(1)}$). By applying lemma 2.5.6 to the sequence $\lambda_\xi = \omega_{\phi(\xi)}$ for $\xi \in (0, \beta]$ and $\lambda_0 = 0$ and by lemmas 2.5.6, 2.3.8 and 2.3.9 we obtain

$$\begin{aligned} C_{p,0}([1, \omega_\gamma]) &\overset{2}{\sim} C_{p,0}([1, \beta]) \times \Pi^*_{\xi<\beta} C_{p,0}([1, \omega_{\phi(\xi+1)} - \omega_{\phi(\xi)}]) \\ &\overset{1}{\sim} C_{p,0}([1, \beta]) \times C_{p,0}([1, \omega_{\phi(1)}]) \times \Pi^*_{1\leq\xi<\beta} C_{p,0}([1, \omega_{\phi(\xi+1)}]) \\ &\overset{2}{\sim} C_{p,0}([1, \beta + \omega_{\phi(1)}]) \times \Pi^*_{1\leq\xi<\beta} C_{p,0}([1, \omega_{\phi(\xi+1)}]) \\ &= C_{p,0}([1, \omega_{\phi(1)}]) \times \Pi^*_{1\leq\xi<\beta} C_{p,0}([1, \omega_{\phi(\xi+1)}]) \\ &\overset{1}{\sim} \Pi^*_{\xi<\beta} C_{p,0}([1, \omega_{\phi(\xi+1)}]). \end{aligned}$$

We applied lemma 2.5.5 to get the second and fifth equivalence and lemmas 2.3.8 and 2.3.9 for the third one. □

2.5.10 LEMMA: *Let α be an initial ordinal and γ an ordinal such that $\overline{\gamma} \leq \overline{\alpha}$, $\gamma \geq 2$. Then $C_{p,0}([1, \alpha^\gamma]) \overset{4}{\sim} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\gamma])$.*

PROOF: First suppose that γ is a successor, say $\gamma = \beta + 1$. By lemmas 2.5.6 (applied to the sequence $\lambda_\xi = \alpha^\beta \cdot \xi$, $\xi \in [0, \alpha]$), 2.5.5, 2.3.8, 2.3.9 and the fact that $\alpha + \alpha^\beta = \alpha^\beta$ if $\beta \geq 2$ (because α^β is a prime component larger than α), we have $C_{p,0}([1, \alpha^\gamma]) \overset{4}{\sim} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\beta])$. By lemma 2.5.4 we now have

$$\Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\gamma]) \overset{4}{\sim} \Pi^*_{\overline{\alpha}} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\beta]) = \Pi^*_{\overline{\alpha}^2} C_{p,0}([1, \alpha^\beta]).$$

Since $\overline{\alpha} = \overline{\alpha}^2$ (page 52), it follows that $C_{p,0}([1, \alpha^\gamma]) \overset{16}{\sim} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\gamma])$, which is as desired.

Now let γ be a limit ordinal. Find $M \subset [2, \gamma)$ as in lemma 2.5.8. Then

$$\begin{aligned} C_{p,0}([1, \alpha^\gamma]) &\overset{4}{\sim} \Pi^*_{\mu\in M} C_{p,0}([1, \alpha^\mu]) && \text{(lemma 2.5.8)} \\ &\overset{16}{\sim} \Pi^*_{\mu\in M} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\mu]) && \text{(by the above and lemma 2.5.4)} \\ &= \Pi^*_{\overline{\alpha}} \Pi^*_{\mu\in M} C_{p,0}([1, \alpha^\mu]) \\ &\overset{4}{\sim} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha^\gamma]) && \text{(lemma 2.5.8).} \ \square \end{aligned}$$

2.5.11 LEMMA: *Let α be a singular ordinal. Then $C_{p,0}([1, \alpha]) \overset{4^5}{\sim} \Pi^*_{\overline{\alpha}} C_{p,0}([1, \alpha])$.*

PROOF: Put $\alpha=\omega_\gamma$. By lemma 2.5.9, there are $\beta<\alpha$ and a strictly increasing β-sequence ϕ with limit γ, such that $\omega_{\phi(\xi)}\geq\beta$ for every $\xi<\beta$ and

$$C_{p,0}([1,\alpha])\overset{4}{\sim}\Pi^*_{\xi<\beta}C_{p,0}([1,\omega_{\phi(\xi+1)}]). \qquad (*)$$

Fix $\xi<\beta$. Notice that $\omega_{\phi(\xi+1)}\cdot\alpha=\alpha$ (proposition 2.1.12). By lemma 2.5.6, applied to the sequence $\lambda_\eta=\omega_{\phi(\xi)}\cdot\eta$ $(\eta<\alpha)$, we obtain

$$\begin{aligned}C_{p,0}([1,\alpha])&\overset{2}{\sim}C_{p,0}([1,\alpha])\times\Pi^*_{\eta<\alpha}C_{p,0}([1,\omega_{\phi(\xi+1)}\cdot(\eta+1)-\omega_{\phi(\xi+1)}\cdot\eta])\\&=C_{p,0}([1,\alpha])\times\Pi^*_{\bar\alpha}C_{p,0}([1,\omega_{\phi(\xi+1)}]).\end{aligned}$$

We now have

$$\begin{aligned}\Pi^*_{\bar\beta}C_{p,0}([1,\alpha])&\overset{2}{\sim}\Pi^*_{\xi<\beta}\Pi^*_{\eta<\alpha}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\bar\beta}C_{p,0}([1,\alpha])\\&=\Pi^*_{\eta<\alpha}\Pi^*_{\xi<\beta}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\bar\beta}C_{p,0}([1,\alpha])\\&\overset{4}{\sim}\Pi^*_{\eta<\alpha}C_{p,0}([1,\alpha])\times\Pi^*_{\bar\beta}C_{p,0}([1,\alpha])\\&\overset{1}{\sim}\Pi^*_{\bar\alpha}C_{p,0}([1,\alpha]).\end{aligned}$$

We applied lemma 2.5.4 to get the first equivalence, and (*) and lemma 2.5.4 to get the third one. The last equivalence follows from lemma 2.5.5 and the fact that $\bar\beta<\bar\alpha$.

Let $B=\{\xi<\beta:\xi\text{ is a limit ordinal}\}\cup\{0\}$. Notice that for every $i\in\mathbb{N}$ and $\xi\in B$, $\overline{\omega_{\phi(\xi+i-1)}\cdot\beta}=\overline{\omega_{\phi(\xi+i-1)}}<\overline{\omega_{\phi(\xi+i)}}$ (by the choice of the sequence ϕ), hence $\omega_{\phi(\xi+i-1)}\cdot\beta<\omega_{\phi(\xi+i)}$, so $\omega_{\phi(\xi+i-1)}\cdot\beta+\omega_{\phi(\xi+i)}=\omega_{\phi(\xi+i)}$ (by theorems 2.1.23 and 2.1.15). Then

$$\begin{aligned}C_{p,0}([1,\alpha])&\overset{4}{\sim}\Pi^*_{\xi<\beta}C_{p,0}([1,\omega_{\phi(\xi+1)}])\\&=\Pi^*_{\xi\in B,\,i\in\mathbb{N}}C_{p,0}([1,\omega_{\phi(\xi+i)}])\\&\overset{1}{\sim}\Pi^*_{\xi\in B}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\xi\in B,\,i\geq2}C_{p,0}([1,\omega_{\phi(\xi+i)}])\\&=\Pi^*_{\xi\in B}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\xi\in B,\,i\geq2}C_{p,0}([1,\omega_{\phi(\xi+i-1)}\cdot\beta+\omega_{\phi(\xi+i)}])\\&\overset{2}{\sim}\Pi^*_{\xi\in B}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\xi\in B,\,i\geq2}C_{p,0}([1,\omega_{\phi(\xi+i-1)}\cdot\beta])\times\\&\quad\times\Pi^*_{\xi\in B,\,i\geq2}C_{p,0}([1,\omega_{\phi(\xi+i)}])\\&\overset{4}{\sim}\Pi^*_{\xi\in B}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\xi\in B,\,i\geq2}\Pi^*_{\bar\beta}C_{p,0}([1,\omega_{\phi(\xi+i-1)}])\times\\&\quad\times\Pi^*_{\xi\in B,\,i\geq2}C_{p,0}([1,\omega_{\phi(\xi+i)}])\\&=\Pi^*_{\xi\in B}C_{p,0}([1,\omega_{\phi(\xi+1)}])\times\Pi^*_{\bar\beta}\Pi^*_{\xi\in B,\,i\in\mathbb{N}}C_{p,0}([1,\omega_{\phi(\xi+i)}])\times\\&\quad\times\Pi^*_{\xi\in B,\,i\geq2}C_{p,0}([1,\omega_{\phi(\xi+i)}])\end{aligned}$$

$$\begin{aligned}
&\overset{1}{\sim} \Pi^*_{\xi\in B,\, i\in\mathbb{N}} C_{p,0}([1,\omega_{\phi(\xi+i)}]) \times \Pi^*_{\beta}\Pi^*_{\xi\in B,\, i\in\mathbb{N}} C_{p,0}([1,\omega_{\phi(\xi+i)}]) \\
&= \Pi^*_{\beta}\Pi^*_{\xi\in B,\, i\in\mathbb{N}} C_{p,0}([1,\omega_{\phi(\xi+i)}]) \\
&= \Pi^*_{\beta}\Pi^*_{\xi<\beta} C_{p,0}([1,\omega_{\phi(\xi+1)}]) \\
&\overset{4}{\sim} \Pi^*_{\beta} C_{p,0}([1,\alpha]).
\end{aligned}$$

Here we applied (*) for the first equivalence, lemma 2.5.5 for the third, lemmas 2.3.8, 2.3.9, 2.5.4 and 2.5.5 for the fifth, and lemmas 2.5.4 and 2.5.7 for the sixth, respectively. To get the eighth and ninth equivalence we used lemma 2.5.5. Finally we used (*) for the last equivalence. We conclude that $C_{p,0}([1,\alpha]) \overset{4\,5}{\sim} \Pi^*_{\alpha} C_{p,0}([1,\alpha])$. □

2.5.12 LEMMA: *Let ξ be an initial ordinal, $\alpha\in[\xi,\xi^2]$, say $\alpha=\xi\cdot\alpha_1+\beta$ with $1\le\alpha_1\le\xi$ and $\beta<\xi$. Then $C_{p,0}([1,\alpha])\sim\Pi^*_{\overline{\alpha_1}} C_{p,0}([1,\xi])$.*

PROOF: First notice that by lemmas 2.3.8 and 2.3.9

$$C_{p,0}([1,\alpha])\sim C_{p,0}([1,\xi\cdot\alpha_1+\beta])\sim C_{p,0}([1,\beta+\xi\cdot\alpha_1])\sim C_{p,0}([1,\xi\cdot\alpha_1]).$$

Then by lemma 2.5.7 we have $C_{p,0}([1,\xi\cdot\alpha_1])\sim\Pi^*_{\overline{\alpha_1}} C_{p,0}([1,\xi])$. This finishes the proof of the lemma. □

At this moment we are able to prove the announced classification. The following theorem states it. The reader should compare it with theorem 2.5.1.

2.5.13 THEOREM: *Let α and β be ordinals.*

If α and β have different power, then

(*a*) *$C_p([1,\alpha])$ and $C_p([1,\beta])$ are not linearly homeomorphic.*

If α and β have the same power and ξ is the initial ordinal of that power, then

(*b*) *If $\xi=\omega$, or ξ is a singular ordinal or both $\alpha,\beta\ge\xi^2$, then $C_p([1,\alpha])\sim C_p([1,\beta])$ if and only if $\max(\alpha,\beta)<[\min(\alpha,\beta)]^{\omega}$.*

(*c*) *If ξ is an uncountable regular ordinal and $\alpha,\beta\in[\xi,\xi^2]$, fix ordinals $\alpha_1,\beta_1\le\xi$ and $\gamma,\delta<\xi$ such that $\alpha=\xi\cdot\alpha_1+\gamma$ and $\beta=\xi\cdot\beta_1+\delta$. Then $C_p([1,\alpha])\sim C_p([1,\beta])$ if and only if $\overline{\alpha_1}=\overline{\beta_1}$.*

(*d*) *If ξ is an uncountable regular ordinal, $\alpha<\xi^2$ and $\beta\ge\xi^2$, then $C_p([1,\alpha])$ and $C_p([1,\beta])$ are not linearly homeomorphic.*

PROOF: Suppose (a) or (d) does not hold for some ordinals α and β. Then by corollary 1.2.21 it also does not hold for $C_0([1,\alpha])$ and $C_0([1,\beta])$. This contradicts theorem

2.5.1 (a) or (d).

We now prove (b). If $C_p([1, \alpha]) \sim C_p([1, \beta])$, then $C_0([1, \alpha]) \sim C_0([1, \beta])$ (corollary 1.2.21). So by theorem 2.5.1 $\max(\alpha, \beta) < [\min(\alpha, \beta)]^\omega$.

We now prove the converse implication. Without loss of generality we may assume that $\alpha \leq \beta$, so suppose that $\beta < \alpha^\omega$.

Case 1: $\xi = \omega$.

Then we can apply theorem 2.4.7 (b).

Case 2: $\alpha, \beta \geq \xi^2$.

By lemma 2.1.5 $\alpha = \xi^\gamma \cdot \lambda + \delta$ for some $\gamma \geq 2$ with $\gamma \leq \alpha$, $1 \leq \lambda < \xi$ and $\delta < \xi^\gamma$. Notice that then $\bar{\gamma} \leq \bar{\alpha} = \bar{\xi}$ and $\alpha < \xi^{\gamma+1}$, thus $\beta < \alpha^\omega \leq \xi^{(\gamma+1)\omega} = \xi^{\gamma\omega}$ (lemma 2.1.19). This implies that $\beta = \xi^{\gamma_i} \cdot \mu + \varepsilon$ for some $i \in \mathbb{N}$, $\mu < \xi^\gamma$ with $\mu > 0$ and $\varepsilon < \xi^{\gamma_i}$ (corollary 2.1.3). Thus

$$\begin{aligned} C_p([1, \alpha]) &\sim C_{p,0}([1, \xi^\gamma \cdot \lambda + \delta]) && \text{(lemma 2.3.9)} \\ &\sim C_{p,0}([1, \delta + \xi^\gamma \cdot \lambda]) && \text{(lemma 2.3.8)} \\ &= C_{p,0}([1, \xi^\gamma \cdot \lambda]) && (\xi^\gamma \text{ is a prime component and } \lambda \neq 0) \end{aligned}$$

In the same way

$$C_p([1, \beta]) \sim C_{p,0}([1, \xi^{\gamma_i} \cdot \mu]).$$

CLAIM 1: For every $p \in \mathbb{N}$ and $1 \leq \nu < \xi^\gamma$ we have $C_{p,0}([1, \xi^{\gamma p} \cdot \nu]) \sim \Pi^*_\xi C_{p,0}([1, \xi^\gamma])$.

First suppose $p \in \mathbb{N}$ and $\nu = 1$. We prove by induction on p that

$$C_{p,0}([1, \xi^{\gamma p}]) \overset{4^{p+3}}{\sim} \Pi^*_\xi C_{p,0}([1, \xi^\gamma]). \qquad (*)$$

The case $p = 1$ follows from lemma 2.5.10. So let $p > 1$. We then have

$$\begin{aligned} C_{p,0}([1, \xi^{\gamma p}]) &= C_{p,0}([1, \xi^{\gamma(p-1)} \cdot \xi^\gamma]) \\ &\overset{4}{\sim} \Pi^*_{\xi^\gamma} C_{p,0}([1, \xi^{\gamma(p-1)}]) && \text{(lemma 2.5.7)} \\ &= \Pi^*_\xi C_{p,0}([1, \xi^{\gamma(p-1)}]) && \text{(proposition 2.1.13)} \\ &\overset{4^{p+2}}{\sim} \Pi^*_\xi \Pi^*_\xi C_{p,0}([1, \xi^\gamma]) && \text{(by induction and lemma 2.5.4)} \\ &= \Pi^*_\xi C_{p,0}([1, \xi^\gamma]) && \text{(page 52).} \end{aligned}$$

Now suppose $p \in \mathbb{N}$ and $1 < \nu < \omega$. Then by induction on ν,

$$\begin{aligned} C_{p,0}([1, \xi^{\gamma p} \cdot \nu]) &\sim C_{p,0}([1, \xi^{\gamma p} \cdot (\nu - 1)]) \times C_{p,0}([1, \xi^{\gamma p}]) && \text{(lemmas 2.3.8 and 2.3.9)} \\ &\sim \Pi^*_\xi C_{p,0}([1, \xi^\gamma]) && \text{(by induction).} \end{aligned}$$

Finally suppose $p \in \mathbb{N}$ and $\nu = \mu + n$ with $\mu \neq 0$ a limit ordinal and n finite. Then

$$\begin{aligned}
C_{p,0}([1, \xi^{\gamma_p} \cdot \nu]) &\sim C_{p,0}([1, \xi^{\gamma_p} \cdot \mu]) \times C_{p,0}([1, \xi^{\gamma_p} \cdot n]) && \text{(lemmas 2.3.8 and 2.3.9)} \\
&\sim \Pi^*_{\bar{\mu}} C_{p,0}([1, \xi^{\gamma_p}]) \times C_{p,0}([1, \xi^{\gamma_p} \cdot n]) && \text{(lemma 2.5.7)} \\
&\sim \Pi^*_{\bar{\mu}} C_{p,0}([1, \xi^{\gamma_p}]) \\
&\sim \Pi^*_{\bar{\mu}} \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi^{\gamma}]) && \text{(lemma 2.5.4 and (*))} \\
&= \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi^{\gamma}]) && (\bar{\mu} \cdot \bar{\xi} = \bar{\xi}),
\end{aligned}$$

and the claim is proved.

By the claim we immediately get

$$C_{p,0}([1, \beta]) \sim C_{p,0}([1, \xi^{\gamma_i} \cdot \mu]) \sim \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi^{\gamma}])$$

and

$$C_{p,0}([1, \alpha]) \sim C_{p,0}([1, \xi^{\gamma} \cdot \lambda]) \sim \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi^{\gamma}]),$$

which proves case 2.

Case 3: ξ is singular and $\alpha \in [\xi, \xi^2]$.

CLAIM 2: If $\alpha \in [\xi, \xi^2]$, then $C_{p,0}([1, \alpha]) \sim C_{p,0}([1, \xi])$.

Indeed, by lemma 2.5.12 we have

$$C_{p,0}([1, \alpha]) \sim \Pi^*_{\mathfrak{m}} C_{p,0}([1, \xi])$$

with $\mathfrak{m} \leq \bar{\xi}$. By lemma 2.5.11, $C_{p,0}([1, \xi]) \overset{45}{\sim} \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi])$, so

$$\begin{aligned}
C_{p,0}([1, \alpha]) &\sim \Pi^*_{\mathfrak{m}} C_{p,0}([1, \xi]) \\
&\sim \Pi^*_{\mathfrak{m}} \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi]) && \text{(lemma 2.5.4)} \\
&= \Pi^*_{\bar{\xi}} C_{p,0}([1, \xi]) && (\mathfrak{m} \cdot \bar{\xi} = \bar{\xi}) \\
&\sim C_{p,0}([1, \xi])
\end{aligned}$$

This proves claim 2.

CLAIM 3: If $\beta \geq \xi^2$ then $C_{p,0}([1, \beta]) \sim C_{p,0}([1, \xi])$.

Indeed, notice that $\beta < \alpha^{\omega} \leq (\xi^2)^{\omega}$. So by case 2, $C_{p,0}([1, \beta]) \sim C_{p,0}([1, \xi^2])$. But by claim 2, $C_{p,0}([1, \xi^2]) \sim C_{p,0}([1, \xi])$, which proves claim 3.

By claims 2 and 3 we finished the proof of case 3, and therefore also the proof of (b).

For (c) first suppose that $C_p([1, \alpha]) \sim C_p([1, \beta])$. Then $C_0([1, \alpha]) \sim C_0([1, \beta])$

(corollary 1.2.21) and thus $\overline{\alpha_1} = \overline{\beta_1}$ (theorem 2.5.1 (c)).

Now suppose $\overline{\alpha_1} = \overline{\beta_1}$. By lemma 2.5.12

$$C_{p,0}([1, \alpha]) \sim \Pi^*_{\overline{\alpha_1}} C_{p,0}([1, \xi]) = \Pi^*_{\overline{\beta_1}} C_{p,0}([1, \xi]) \sim C_{p,0}([1, \beta])$$

and (c) is proved. □

The last theorem gives a complete isomorphical classification of the spaces $C_p(X)$, where X is a compact ordinal space. As announced in the introduction of this section, Kislyakov gave the same classification for the spaces $C_0(X)$ (with X a compact ordinal space). However, he made a mistake in his proof. We now will point out his mistake, and indicate how it can be corrected.

Kislyakov states the following: "Let $\alpha = \omega_\gamma$. Since α is singular, it follows that $\gamma < \alpha$ and ...." (cf. [34, lemma 3.3]). But in example 2.1.14 we gave an example of a singular ordinal ω_γ such that $\gamma = \omega_\gamma$.

An examination of our proofs tells us that if lemmas 2.5.4, 2.5.5 and 2.5.6 hold for function spaces endowed with the topology of uniform convergence, then all the other lemmas and theorems also hold for function spaces with this topology. Kislyakov proved lemmas 2.5.4 and 2.5.6 for those function spaces (cf. resp. lemma 1.2 and lemma 3.1 in [34]). In addition, lemma 2.5.5 is very easy to prove for function spaces endowed with the topology of uniform convergence. So our proof can be copied to get a correct proof of the classification of Kislyakov. It turns out that the proof one gets in this way differs from the proof of Kislyakov at two places. First of all corollary 3.3 in [34] has to be stated in a more general form (it becomes our lemma 2.5.9 for function spaces endowed with the topology of uniform convergence) and second, the proof of lemma 3.3 of [34] (which is our lemma 2.5.11 for function spaces endowed with the topology of uniform convergence) has to be fixed (the proof for the case $\gamma < \omega_\gamma$ remains the same but the case $\gamma = \omega_\gamma$ has to be added).

Finally we remark that Gulko and Oskin also proved theorem 2.5.1 (b) and (d) (in [28]), independently from Kislyakov. We were inspired by [34] because [28] contains no proofs. The other results in this section are new and were never published.

§2.6. σ-compact ordinals

In this section we give a complete isomorphical classification of the spaces $C_p(X)$

and $C_0(X)$ where $X=[1, \alpha)$, for ordinals α cofinal with ω. Notice that these spaces are exactly the non-compact spaces which are a countable union of compact ordinal spaces. Therefore we call such an ordinal a ***σ-ordinal***. If α is also a prime component or an initial ordinal, we call it a ***σ-prime component*** or a ***σ-initial ordinal***, respectively.

2.6.1 LEMMA: *If α is a σ-ordinal, then every closed and bounded subset of $[1, \alpha)$ is compact.*

PROOF: Let $(\alpha_n)_n$ be a strictly increasing sequence of ordinals with limit α. Let A be a closed and bounded subset of $[1, \alpha)$. Then there is $n \in \mathbb{N}$ such that $A \subset [1, \alpha_n]$. For if not, then A would contain a closed discrete subset, which is not possible because A is bounded. Since A is closed in $[1, \alpha_n]$ it is compact. □

2.6.2 REMARK: From lemma 2.6.1 and corollary 1.2.21 we have for σ-ordinals α and β that a linear homeomorphism $\phi: C_p([1, \alpha)) \to C_p([1, \beta))$ considered as a map from $C_0([1, \alpha))$ to $C_0([1, \beta))$ is also a linear homeomorphism.

Furthermore, let β be an ordinal with $cf(\beta) > \omega$. By the methods of [24, Ex 3.1.27] it easily follows that every continuous function $f: [1, \beta) \to \mathbb{R}$ is eventually constant. But this implies that $[1, \beta)$ is pseudocompact. By this observation it follows that lemma 2.6.1 does not hold for β and that α is a σ-ordinal if and only if $[1, \alpha)$ is a non-compact non-pseudocompact space

The following lemma is the key lemma in the proof of the classification mentioned above.

2.6.3 LEMMA: *Let X and Y be spaces such that $X = X_1 \oplus X_2 \oplus X_3$ and $Y = Y_1 \oplus Y_2 \oplus Y_3$. Suppose $\phi: C_0(X) \to C_0(Y)$ is a linear homeomorphism such that $\operatorname{supp} X_1 \subset Y_1$ and $\operatorname{supp} Y_2 \subset X_1 \oplus X_2$. Then there is a linear embedding $\theta: C_0(Y_2) \to C_0(X_2)$.*

PROOF: For each $f \in C_0(Y_2)$ we define $f^* \in C_0(Y)$ by $f^*(y) = f(y)$ if $y \in Y_2$ and $f^*(y) = 0$ elsewhere. In a similar way we define for every $g \in C_0(X_2)$, $g^+ \in C_0(X)$. Define $\theta: C_0(Y_2) \to C_0(X_2)$ by $\theta(f) = \phi^{-1}(f^*) \mid X_2$ and $\psi: C_0(X_2) \to C_0(Y_2)$ by $\psi(g) = \phi(g^+) \mid Y_2$.

Then θ and ψ are continuous linear functions. Furthermore for every $h \in C_0(Y_2)$, we have $\psi(\theta(h)) = h$. Indeed, assume to the contrary that $\phi(\theta(h)^+) \mid Y_2 \neq h^* \mid Y_2$. Then $\theta(h)^+ \mid (X_1 \oplus X_2) \neq \phi^{-1}(h^*) \mid (X_1 \oplus X_2)$ since $X_1 \oplus X_2$ is a neighborhood of $\operatorname{supp} Y_2$

and ϕ is effective (corollary 1.2.15 (a)). Now $h^* \equiv 0$ on Y_1, so $\phi^{-1}(h^*) \equiv 0$ on X_1, since Y_1 is a neighborhood of $\operatorname{supp} X_1$ and ϕ^{-1} is effective. Furthermore $\theta(h)^+ \equiv 0$ on X_1, so that $\theta(h)^+ = \phi^{-1}(h^*)$ on X_1. This implies $\theta(h)^+ | X_2 \neq \phi^{-1}(h^*) | X_2$, which is impossible because both the left-hand side and the right-hand side are equal to $\theta(h)$. This is a contradiction and we conclude that θ is a linear embedding. □

2.6.4 COROLLARY: *Let X and Y be spaces such that $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Suppose $\phi: C_0(X) \to C_0(Y)$ is a linear homeomorphism such that $\operatorname{supp} Y_1 \subset X_1$. Then there is a linear embedding $\theta: C_0(Y_1) \to C_0(X_1)$.*

PROOF: Take $X_1 = Y_1 = \emptyset$ in lemma 2.6.3. □

Notice that lemma 2.6.3 and corollary 2.6.4 also hold for the spaces $C_p(X)$ and $C_p(Y)$.

The strategy of the proof of the classification is as follows: First we define a class of spaces, and we prove that for every σ-ordinal α there is a space Y in this class such that $C_p([1, \alpha)) \sim C_p(Y)$ (lemma 2.6.6). Then we prove that if X and Y are two spaces in this class, then $C_p(X) \sim C_p(Y)$ if and only if $C_0(X) \sim C_0(Y)$ if and only if $X = Y$ (corollary 2.6.15 and lemma 2.6.16). From these results we then easily derive our classification (theorem 2.6.17).

For initial ordinals α and β with $\alpha \geq \beta \geq \omega$ we define the following classes of spaces:

Case 1: If α is singular or ω and β is singular or ω then

$$\mathcal{A}^{(\alpha,\beta)} = \{[1, \omega^\mu] \oplus [1, \omega^\tau) : \mu \text{ a prime component, } \tau \text{ a } \sigma\text{-prime component or } \tau = 1, \mu \geq \tau \geq 1, \overline{\omega^\mu} = \overline{\alpha}, \overline{\omega^\tau} = \overline{\beta}\},$$

Case 2: If α is uncountable regular and β is singular or ω then

$$\mathcal{A}^{(\alpha,\beta)} = \{[1, \omega^\mu] \oplus [1, \omega^\tau) : \mu \text{ a prime component, } \tau \text{ a } \sigma\text{-prime component or } \tau = 1, \omega^\mu > \alpha^2, \overline{\omega^\mu} = \overline{\alpha}, \overline{\omega^\tau} = \overline{\beta}\}$$
$$\cup \{[1, \alpha \cdot \xi] \oplus [1, \omega^\tau) : \tau \text{ a } \sigma\text{-prime component or } \tau = 1, \xi \text{ initial}, 1 \leq \xi \leq \alpha, \overline{\omega^\tau} = \overline{\beta}\}.$$

Case 3: If α is singular or ω and β is uncountable regular then

$$\mathcal{A}^{(\alpha,\beta)} = \{[1, \omega^\mu] \oplus [1, \omega^\tau) : \mu \text{ a prime component, } \tau \text{ a } \sigma\text{-prime component, } \omega^\tau > \beta^2, \overline{\omega^\mu} = \overline{\alpha}, \overline{\omega^\tau} = \overline{\beta}\}$$
$$\cup \{[1, \omega^\mu] \oplus [1, \beta \cdot \eta) : \mu \text{ a prime component, } \eta = \rho \cdot \omega \text{ with } \rho \text{ initial or } \eta \text{ } \sigma\text{-initial}, \omega \leq \eta \leq \beta, \overline{\omega^\mu} = \overline{\alpha}\}.$$

Case 4: If α and β are uncountable regular then

$$\begin{aligned}\mathcal{A}^{(\alpha,\beta)} = \{&[1,\omega^\mu]\oplus[1,\omega^\tau): \mu \text{ a prime component, } \tau \text{ a } \sigma\text{-prime component, } \\ &\mu\geq\tau,\ \omega^\mu>\alpha^2,\ \omega^\tau>\beta^2,\ \overline{\omega^\mu}=\bar{\alpha},\ \overline{\omega^\tau}=\bar{\beta}\}\\ \cup\ \{&[1,\alpha\cdot\xi]\oplus[1,\omega^\tau): \tau \text{ a } \sigma\text{-prime component, } \xi \text{ initial, } 1\leq\xi\leq\alpha,\\ &\alpha\cdot\xi\geq\omega^\tau,\ \omega^\tau>\beta^2,\ \overline{\omega^\tau}=\bar{\beta}\}\\ \cup\ \{&[1,\omega^\mu]\oplus[1,\beta\cdot\eta): \mu \text{ a prime component, } \eta=\rho\cdot\omega \text{ with } \rho \text{ initial or } \eta\\ &\sigma\text{-initial, } \omega\leq\eta\leq\beta,\ \omega^\mu\geq\beta\cdot\eta,\ \overline{\omega^\mu}=\bar{\alpha}\}\\ \cup\ \{&[1,\alpha\cdot\xi]\oplus[1,\beta\cdot\eta): \xi \text{ initial, } \eta=\rho\cdot\omega \text{ with } \rho \text{ initial or } \eta\ \sigma\text{-initial,}\\ &1\leq\xi\leq\alpha,\ \omega\leq\eta\leq\beta,\ \alpha\cdot\xi\geq\beta\cdot\eta\}.\end{aligned}$$

Now let $\beta\geq\omega$ be an initial ordinal.

Case 5: If β is singular or $\beta=\omega$ then

$$\mathcal{B}^{(\beta)}=\{[1,\omega^\tau): \tau \text{ a } \sigma\text{–prime component or } \tau=1,\ \overline{\omega^\tau}=\bar{\beta}\}.$$

Case 6: If β is uncountable regular then

$$\begin{aligned}\mathcal{B}^{(\beta)} = \{&[1,\omega^\tau): \tau \text{ a } \sigma\text{–prime component, } \omega^\tau>\beta^2,\ \overline{\omega^\tau}=\bar{\beta}\}\\ \cup\ \{&[1,\beta\cdot\eta): \eta=\rho\cdot\omega \text{ with } \rho \text{ initial or } \eta\ \sigma\text{–initial, } \omega\leq\eta\leq\beta\}.\end{aligned}$$

Now let

$$\mathcal{A}=\bigcup\{\mathcal{A}^{(\alpha,\beta)}: (\alpha,\beta) \text{ as in case 1, 2, 3 or 4}\},$$

and

$$\mathcal{B}=\bigcup\{\mathcal{B}^{(\beta)}: \beta \text{ as in case 5 or 6}\}.$$

The class of spaces that we are currently interested in is $\mathcal{A}\cup\mathcal{B}$. Notice that whenever $X=[1,\phi]\oplus[1,\psi)\in\mathcal{A}$, then $\phi\geq\psi$.

For every space $X\in\mathcal{A}\cup\mathcal{B}$ we need to fix a certain decomposition. First we will assign to certain ordinals μ a fixed sequence $(\mu_i)_i$ of ordinals. If $\mu=1$, put $\mu_i=0$ for each $i\in\mathbb{N}$. If $\mu=\tau\cdot\omega$ for some τ, put $\mu_i=\tau\cdot i$ for each $i\in\mathbb{N}$, and if μ is a σ-ordinal not of the form $\tau\cdot\omega$, let $(\mu_i)_i$ be a strictly increasing sequence of ordinals such that $\mu_i\to\mu$ and $1\leq\mu_i<\mu$ for each $i\in\mathbb{N}$. We now define the desired decompositions:

If $X=[1,\phi]\oplus[1,\omega^\tau)\in\mathcal{A}$, then $X=[1,\phi]\oplus[1,\omega^{\tau_1}]\oplus[1,\omega^{\tau_2}]\oplus\cdots$ (this is true because for every i, ω^{τ_i} is a prime component).

If $X=[1,\phi]\oplus[1,\beta\cdot\eta)\in\mathcal{A}$, then $X=[1,\phi]\oplus[1,\beta\cdot\eta_1]\oplus[1,\beta\cdot\eta_2]\oplus\cdots$

If $X=[1,\omega^\tau)\in\mathcal{B}$, then $X=[1,\omega^{\tau_1}]\oplus[1,\omega^{\tau_2}]\oplus\cdots$

If $X=[1,\beta\cdot\eta)\in\mathcal{B}$, then $X=[1,\beta\cdot\eta_1]\oplus[1,\beta\cdot\eta_2]\oplus\cdots$

If for $X\in\mathcal{A}\cup\mathcal{B}$ we write $X=\bigoplus_{i=1}^{\infty}X_i$, then we implicitly mean that the X_i are as above.

Now we are going to prove that for every σ-ordinal ϕ there is a space $Y\in\mathcal{A}\cup\mathcal{B}$ and

a decomposition $\oplus_{i=1}^{\infty} X_i$ of $[1, \phi)$ such that $C_p(X_i) \sim C_p(Y_i)$. We first need the following

2.6.5 LEMMA: *Let $\beta \geq \omega$ be an initial ordinal and τ a successor or a σ-ordinal, such that $\overline{\omega^\tau} = \bar{\beta}$.*

(a) If $\beta = \omega$, β is singular, or $\omega^{\tau'} \geq \beta^2$ and if τ is not a prime component, then there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of $[1, \omega^\tau)$, such that for every i, $C_p(X_i) \sim C_p([1, \omega^{\tau' \cdot i}])$. In particular $C_p([1, \omega^\tau)) \sim C_p([1, \omega^{\tau' \cdot \omega}))$.

(b) If β is uncountable regular and $\omega^{\tau'} \in [\beta, \beta^2)$ then one of the following holds:

(i) $\omega^\tau = \beta \cdot \eta$ with η σ-initial and $\omega \leq \eta \leq \beta$, or

(ii) there is an initial ordinal η such that $\omega \leq \eta \leq \beta$ and there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of $[1, \omega^\tau)$, such that for every i, $C_p(X_i) \sim C_p([1, \beta \cdot \eta \cdot i])$. In particular $C_p([1, \omega^\tau)) \sim C_p([1, \beta \cdot \eta \cdot \omega))$. Furthermore $\beta \cdot \eta \cdot \omega \leq \omega^\tau$.

PROOF: First notice that $[1, \omega^\tau) = \oplus_{i=1}^{\infty} [1, \omega^{\tau_i}]$, where $\tau_i = \nu$ if $\tau = \nu + 1$ and $(\tau_i)_i$ is a strictly increasing sequence (not necessary equal to the fixed sequence associated with τ) with limit τ if τ is a σ-ordinal. Both in (a) and (b)(ii) we will get $X_i = [1, \omega^{\tau_i}]$.

We first prove (a). Since τ is not a prime component, we have $\tau' < \tau < \tau' \cdot \omega$ (lemma 2.1.19), and we can assume $\tau' \leq \tau_i$ for each i. Now $\omega^{\tau'} \leq \omega^{\tau_i} < \omega^\tau < (\omega^{\tau'})^\omega$. With the help of proposition 2.1.13 it now easily follows that $\overline{\omega^{\tau_i}} = \overline{\omega^{\tau'}} = \bar{\beta}$. Since $\beta = \omega$, β is singular, or $\omega^{\tau'} \geq \beta^2$, we can apply theorem 2.5.13 (b) to obtain

$$C_p([1, \omega^{\tau_i}]) \sim C_p([1, \omega^{\tau'}])$$

By a similar argument

$$C_p([1, \omega^{\tau'}]) \sim C_p([1, \omega^{\tau' \cdot i}]),$$

whence by lemma 2.3.7,

$$C_p([1, \omega^\tau)) = C_p(\bigoplus_{i=1}^{\infty} [1, \omega^{\tau_i}]) \sim C_p(\bigoplus_{i=1}^{\infty} [1, \omega^{\tau' \cdot i}]) = C_p([1, \omega^{\tau' \cdot \omega})).$$

For (b) we distinguish two cases.

Case 1: $\omega^\tau > \beta^2$.

Since β is initial, it is a prime component, so by theorem 2.1.21 $\beta = \omega^\rho$ for some ordinal ρ. Then $\rho \cdot 2 < \tau$ and we can assume that $\rho \cdot 2 \leq \tau_i$ for each i. We conclude that $\omega^{\tau_i} \geq \beta^2$. Since $\omega^{\tau'} < \beta^2$ we have $\tau' < \rho \cdot 2 < \tau$, so $\tau < \rho \cdot \omega$ by lemma 2.1.19. Thus $\beta^2 \leq \omega^{\tau_i} < (\beta^2)^\omega$. By proposition 2.1.13 it now easily follows that $\overline{\omega^{\tau_i}} = \bar{\beta}$, so by theorem 2.5.13 (b)

$$C_p([1, \omega^{\tau_i}]) \sim C_p([1, \beta^2])$$

and similarly

$$C_p([1, \beta^2]) \sim C_p([1, \beta^2 \cdot i]).$$

Consequently,

$$C_p([1, \omega^{\tau})) \sim C_p([1, \beta^2 \cdot \omega)).$$

Since $\beta^2 \cdot \omega$ is the smallest prime component larger than β^2 (lemma 2.1.19), $\omega^{\tau} \geq \beta^2 \cdot \omega$, and so we have established (ii) for $\eta = \beta$.

Case 2: $\omega^{\tau} \leq \beta^2$.

If $\omega^{\tau} = \beta^2$, ω^{τ} satisfies (i), so we may assume $\omega^{\tau} < \beta^2$. There are ordinals $\delta, \eta^* < \beta$ such that $\omega^{\tau} = \beta \cdot \eta^* + \delta$. Since $\delta < \omega^{\tau}$ and ω^{τ} is a prime component, $\delta = 0$. In addition, since β is regular and $cf(\omega^{\tau}) = \omega$, $\eta^* \geq \omega$. If η^* is initial we are done, so suppose η^* is not initial. Let η be the initial ordinal of the same power as η^*. Then $\omega \leq \eta < \eta^*$, hence $\beta \cdot \eta < \beta \cdot \eta^* = \omega^{\tau}$, so we can assume $\beta \cdot \eta \leq \omega^{\tau_i}$ for each i. Write $\omega^{\tau_i} = \beta \cdot \eta_i + \delta_i$ with $\eta_i, \delta_i < \beta$. Then $\delta_i = 0$ since ω^{τ_i} is a prime component, and $\beta \cdot \eta \leq \beta \cdot \eta_i < \beta \cdot \eta^*$, whence $\overline{\eta} = \overline{\eta}_i$. Since $\omega^{\tau_i} = \beta \cdot \eta_i$, $\beta \cdot \eta \in [\beta, \beta^2]$, it follows by theorem 2.5.13 (c) that $C_p([1, \omega^{\tau_i}]) \sim C_p([1, \beta \cdot \eta])$. Since $C_p([1, \beta \cdot \eta]) \sim C_p([1, \beta \cdot \eta \cdot i])$, (ii) can be established as in case 1. □

2.6.6 LEMMA: *Let ϕ be a σ-ordinal. Then there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of $[1, \phi)$ and a space $Y \in \mathcal{A} \cup \mathcal{B}$ such that $C_p(X_i) \sim C_p(Y_i)$ (where Y_i is the i^{th} term in the fixed decomposition of Y). In particular $C_p([1, \phi)) \sim C_p(Y)$ and $C_0([1, \phi)) \sim C_0(Y)$.*

PROOF: By theorems 2.1.16 and 2.1.21 there are ordinals ψ and τ such that $\phi = \psi + \omega^{\tau}$, with $\tau > 0$ and $\psi = 0$ or $\psi \geq \omega^{\tau}$. Notice that τ is a successor or a σ-ordinal. Let α and β be initial such that $\overline{\alpha} = \overline{\psi}$ and $\overline{\beta} = \overline{\omega^{\tau}}$. Notice that $\omega^{\tau'} \neq \beta^2$, because if not then $\omega^{\tau'} = \beta^2 = \omega^{\rho \cdot 2}$ for some prime component ρ which implies that $\tau' = \rho \cdot 2$ is not a prime component.

Case 1: $\psi = 0$, $\tau' = \tau$.

If β is singular, $\beta = \omega$ or $\omega^{\tau} > \beta^2$, we have $[1, \phi) = [1, \omega^{\tau}) \in \mathcal{B}$.

If β is uncountable regular and $\omega^{\tau} \in [\beta, \beta^2)$, then by lemma 2.6.5 we have either $[1, \omega^{\tau}) \in \mathcal{B}$ or there is $Y \in \mathcal{B}$ such that $C_p([1, \phi)) = C_p([1, \omega^{\tau})) \sim C_p(Y)$ with the desired decomposition and such that if $Y = [1, \delta)$ then $\delta \leq \omega^{\tau}$.

Case 2: $\psi = 0$, $\tau' \neq \tau$.

If β is singular, $\beta = \omega$ or $\omega^{\tau'} > \beta^2$, we have by lemma 2.6.5 (a) that there exists a space $Y \in \mathcal{B}$ such that $C_p([1, \phi)) = C_p([1, \omega^{\tau})) \sim C_p(Y)$ with the desired decomposition

($Y \in \mathcal{B}$ because $Y = [1, \omega^{\tau'\cdot\omega})$ and $\tau'\cdot\omega$ is a σ-prime component).

If β is uncountable regular and $\omega^{\tau'} < \beta^2$, by lemma 2.6.5 (b) there is a space $Y \in \mathcal{B}$ such that $C_p([1, \phi)) \sim C_p(Y)$ and which has the desired decomposition.

Case 3: $\psi \geq \omega^\tau$, $\tau' = \tau$.

There is an ordinal $\mu \geq \tau$ such that $\psi' = \omega^\mu$. Notice that $\tau' \leq \mu$. By case 1, there is a space $Y' = [1, \delta) \in \mathcal{B}$ with $\delta \leq \omega^\tau$, such that $C_p([1, \omega^{\tau'})) \sim C_p(Y')$ and which has the desired decomposition.

If α is singular, $\alpha = \omega$ or $\omega^{\mu'} > \alpha^2$, by theorem 2.5.13 (b) $C_p([1, \omega^{\mu'}]) \sim C_p([1, \psi])$ (because $\omega^{\mu'} \leq \omega^\mu \leq \psi < \omega^{\mu+1} < (\omega^{\mu'})^\omega$). Since $\omega^{\mu'} \geq \omega^{\tau'} = \omega^\tau \geq \delta$,

$$[1, \omega^{\mu'}] \oplus [1, \delta) = [1, \omega^{\mu'}] \oplus Y' \in \mathcal{A}.$$

If α is uncountable regular and $\omega^{\mu'} < \alpha^2$ we have to consider two subcases

Subcase 3.1: $\psi \geq \alpha^2$.

Then $\alpha^2 \leq \psi < (\omega^{\mu'})^\omega \leq (\alpha^2)^\omega$, so by theorem 2.5.13 (b), $C_p([1, \psi]) \sim C_p([1, \alpha^2])$. Since $\alpha^2 > \omega^{\mu'} \geq \omega^{\tau'} \geq \delta$, the space $Y = [1, \alpha^2] \oplus Y' \in \mathcal{A}$ and $C_p([1, \phi)) \sim C_p(Y)$ and moreover has the desired decomposition.

Subcase 3.2: $\psi < \alpha^2$.

Then $\psi = \alpha\cdot\xi^* + \delta$ with $1 \leq \xi^* < \alpha$ and $\delta < \alpha$. If we let ξ the initial ordinal with the same power as ξ^*, then by theorem 2.5.13 (c), $C_p([1, \psi]) \sim C_p([1, \alpha\cdot\xi])$. Now let $Y = [1, \alpha\cdot\xi] \oplus Y'$. It is easily seen that if $\alpha\cdot\xi < \delta$, then $Y \approx Y' \in \mathcal{B}$ and otherwise $Y \in \mathcal{A}$.

Case 4: $\psi \geq \omega^\tau$, $\tau' \neq \tau$.

This is a combination of the cases 2 and 3.

Notice that the last remark in the lemma easily follows from lemma 2.3.7 and corollary 1.2.21. □

Now we are going to prove that for every $X, Y \in \mathcal{A} \cup \mathcal{B}$ we have $C_p(X) \sim C_p(Y)$ if and only if $C_0(X) \sim C_0(Y)$ if and only if $X = Y$. For that we first have to do some prepatory work.

2.6.7 LEMMA:

(*a*) *If* $\theta\colon C_0([1, \omega^\mu]) \to C_0([1, \omega^\nu])$ *is a linear embedding with* $\mu, \nu \geq 1$, *then*

(*i*) $\mu < \nu\cdot\omega$, *hence* $\omega^\mu < (\omega^\nu)^\omega$, *and*

(*ii*) *if* μ *is a prime component, then* $\mu \leq \nu$, *hence* $\omega^\mu \leq \omega^\nu$.

(*b*) *Let* α *be an uncountable regular ordinal and* $\xi, \eta \in [1, \alpha]$.

If $\theta\colon C_0([1, \alpha\cdot\xi]) \to C_0([1, \alpha\cdot\eta])$ *is a linear embedding, then* $\overline{\xi} \leq \overline{\eta}$.

PROOF: We first prove (a). Suppose $\mu \geq \nu\cdot\omega$. Then $\omega^\mu \geq (\omega^\nu)^\omega$, so by corollary 2.5.3

$C_0([1, \omega^\nu])$ has smaller linear dimension than $C_0([1, \omega^\mu])$, which contradicts the fact that $\theta: C_0([1, \omega^\mu]) \to C_0([1, \omega^\nu])$ is a linear embedding. This proves (i). For (ii) let μ be a prime component. Since $\mu < \nu \cdot \omega$, by lemma 2.1.19, $\mu \leq \nu' \leq \nu$.

For (b) suppose $\overline{\eta} < \overline{\xi}$. Then $\eta < \xi$, so there is a linear embedding $\phi: C_0([1, \alpha \cdot \eta]) \to C_0([1, \alpha \cdot \xi])$. This gives that $C_0([1, \alpha \cdot \eta])$ and $C_0([1, \alpha \cdot \xi])$ have the same linear dimension, so by theorem 2.5.1 (c) $\overline{\eta} = \overline{\xi}$. Contradiction. □

2.6.8 LEMMA:

(a) Let $X = Z \oplus [1, \omega)$ with Z a compact space, and $Y = \oplus_{i=1}^{\infty} Z_i$ where each Z_i is an infinite compact space. Then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.

(b) Let $X = Z_1 \oplus Z_2$ with Z_1 an infinite compact space. Then $C_0(X)$ is not linearly homeomorphic to $C_0([1, \omega))$.

PROOF: For (a) suppose that $C_0(X)$ is linearly homeomorphic to $C_0(Y)$. Then by corollary 1.2.15 (b) there is $n \in \mathbb{N}$ such that $\operatorname{supp} Z \subset \oplus_{i=1}^{n} Z_i$. Again by corollary 1.2.15 (b) there is $m \in \mathbb{N}$ such that $\operatorname{supp} Z_{n+1} \subset Z \oplus [1, m]$. By lemma 2.6.3, there is a linear embedding $\theta: C_0(Z_{n+1}) \to C_0([1, m]) = \mathbb{R}^m$. Since Z_{n+1} is infinite we have a contradiction, because the algebraic dimension of $C_0(Z_{n+1})$ is infinite.

For (b) suppose that $C_0(X)$ is linearly homeomorphic to $C_0([1, \omega))$. Then by corollary 1.2.15 (b), there is $m \in \mathbb{N}$ such that $\operatorname{supp} Z_1 \subset [1, m]$. By corollary 2.6.4, there is a linear embedding $\theta: C_0(Z_1) \to C_0[1, m] = \mathbb{R}^m$. Again we have a contradiction. □

2.6.9 LEMMA: *Let $X = [1, \xi_1] \oplus [1, \xi_2)$ and $Y = [1, \eta_1] \oplus [1, \eta_2)$ where ξ_2 and η_2 are σ-prime components, $\xi_1 \geq \xi_2$ and $\eta_1 \geq \eta_2$. Then $C_0(X) \sim C_0(Y)$ implies $\overline{\xi_1} = \overline{\eta_1}$ and $\overline{\xi_2} = \overline{\eta_2}$.*

PROOF: Suppose $\overline{\xi_1} < \overline{\eta_1}$ and $C_0(X) \sim C_0(Y)$. By corollary 1.2.15 (b) there is $\delta < \xi_2$ such that $\operatorname{supp} [1, \eta_1] \subset [1, \xi_1 + \delta]$, which implies by corollary 2.6.4 that there is a linear embedding $\theta: C_0([1, \eta_1]) \to C_0([1, \xi_1 + \delta])$. Since $\overline{\xi_1 + \delta} = \overline{\xi_1}$, $\overline{\xi_1 + \delta} < \overline{\eta_1}$ and so $\xi_1 + \delta < \eta_1$. But then there is also a linear embedding $\phi: C_0([1, \xi_1 + \delta]) \to C_0([1, \eta_1])$ and we conclude that $C_0([1, \xi_1 + \delta])$ and $C_0([1, \eta_1])$ have the same linear dimension. This contradicts theorem 2.5.1 (a). By symmetry we conclude that $\overline{\xi_1} = \overline{\eta_1}$.

Now suppose $\overline{\xi_2} < \overline{\eta_2}$. By corollary 1.2.15 (b) there is $\delta < \eta_2$ such that $\operatorname{supp} [1, \xi_1] \subset [1, \eta_1 + \delta]$. Since η_2 is a prime component, $Y \approx [1, \eta_1 + \delta] \oplus [1, \eta_2)$.

CLAIM: There is an ordinal $\tau < \eta_2$, such that $\overline{\tau} > \overline{\xi_2}$.

Indeed, choose ω_α and ω_β such that $\overline{\xi_2} = \overline{\omega_\alpha}$ and $\overline{\eta_2} = \overline{\omega_\beta}$. Notice that $\alpha + 1 \leq \beta$ (theorem 2.1.6). If $\alpha + 1 < \beta$, then $\tau = \omega_{\alpha+1}$ satisfies the claim. If $\alpha + 1 = \beta$, then ω_β is regular (theorem 2.1.10). Since η_2 is a σ-ordinal, it follows that $\omega_\beta < \eta_2$, so let $\tau = \omega_\beta$.

Now choose $\sigma < \xi_2$ such that $\operatorname{supp}[1, \tau] \subset [1, \xi_1] \oplus [1, \sigma]$. By lemma 2.6.3, there is a linear embedding $\theta: C_0([1, \tau]) \to C_0([1, \sigma])$. Since $\overline{\sigma} \leq \overline{\xi_2} < \overline{\tau}$, $\sigma < \tau$. But then there is also a linear embedding $\phi: C_0([1, \sigma]) \to C_0([1, \tau])$ and we may conclude that $C_0([1, \sigma])$ and $C_0([1, \tau])$ have the same linear dimension. This contradicts theorem 2.5.1 (a). By symmetry we conclude that $\overline{\xi_2} = \overline{\eta_2}$. □

2.6.10 LEMMA: *Let α be an initial ordinal.*

(a) Let $X = [1, \omega^\mu] \oplus [1, \beta)$ and $Y = [1, \omega^\sigma] \oplus [1, \gamma)$, where $1 \leq \mu \leq \sigma$ are prime components, $\omega^\mu \geq \beta$ and $\omega^\sigma \geq \gamma$. Then $C_0(X) \sim C_0(Y)$, implies $\mu = \sigma$.

(b) Let $X = [1, \alpha \cdot \xi] \oplus [1, \beta)$ and $Y = [1, \alpha \cdot \eta] \oplus [1, \gamma)$ where α is uncountable regular, $1 \leq \xi \leq \eta < \alpha$, ξ and η are initial, $\alpha \cdot \xi \geq \beta$, and $\alpha \cdot \eta \geq \gamma$. Then $C_0(X) \sim C_0(Y)$ implies $\xi = \eta$.

PROOF: For (a), by corollary 1.2.15 (b) there is $\delta < \beta$ such that $\operatorname{supp}[1, \omega^\sigma] \subset [1, \omega^\mu] \oplus [1, \delta]$. Since ω^μ is a prime component, we have $[1, \omega^\mu] \oplus [1, \delta] \approx [1, \omega^\mu]$. Hence by corollary 2.6.4, there is a linear embedding $\theta: C_0([1, \omega^\sigma]) \to C_0([1, \omega^\mu])$. Then by lemma 2.6.7 (a) we have $\sigma \leq \mu$. Since by assumption $\sigma \geq \mu$, $\sigma = \mu$.

For the proof of (b), let $\delta < \beta$ be such that $\operatorname{supp}[1, \alpha \cdot \eta] \subset [1, \alpha \cdot \xi] \oplus [1, \delta]$. Since $\delta < \alpha \cdot \xi$ we have $\delta + \alpha \cdot \xi = \alpha \cdot \xi$ and so $[1, \alpha \cdot \xi] \oplus [1, \delta] \approx [1, \alpha \cdot \xi]$. But then by corollary 2.6.4 there is a linear embedding $\theta: C_0([1, \alpha \cdot \eta]) \to [1, \alpha \cdot \xi])$. Hence by lemma 2.6.7 (b), $\overline{\eta} \leq \overline{\xi}$. Since η and ξ are initial, $\eta \leq \xi$ and so $\eta = \xi$ (because by assumption $\eta \geq \xi$). □

2.6.11 LEMMA: *Let α be an initial ordinal.*

(a) Let $X = Z_1 \oplus [1, \omega^\delta)$ and $Y = Z_2 \oplus [1, \omega^{\tau \cdot \omega})$, where Z_1 and Z_2 are compact spaces, δ, τ are prime components, $\delta = 1$ or δ is a σ-ordinal, $1 \leq \delta \leq \tau$, and $\overline{\omega^\delta} = \overline{\omega^\tau} = \overline{\alpha}$. If α is singular, $\alpha = \omega$ or $\omega^\delta \geq \alpha^2$, then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.

(b) Let $X = Z_1 \oplus [1, \alpha \cdot \xi)$ and $Y = Z_2 \oplus [1, \alpha \cdot \eta \cdot \omega)$, where Z_1 and Z_2 are compact spaces, ξ is σ-initial, η is initial, and $\omega \leq \xi \leq \eta \leq \alpha$. If α is uncountable regular, then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.

PROOF: For (a) suppose the $C_0(X) \sim C_0(Y)$. Then by lemma 2.6.8 (a), $\delta > 1$. By

corollary 1.2.15 (b) there is $n \in \mathbb{N}$ such that $\operatorname{supp} Z_1 \subset Z_2 \oplus [1, \omega^{\tau \cdot n}]$. Again by corollary 1.2.15 (b) there is δ_i in the fixed sequence associated with δ such that $\operatorname{supp}[1, \omega^{\tau\cdot(n+1)}] \subset Z_1 \oplus [1, \omega^{\delta_i}]$. By lemma 2.6.3 and the fact that $C_0([1, \omega^{\tau\cdot(n+1)}]) \sim C_0([1, \omega^{\tau}])$ (theorem 2.5.13 (b)), there is a linear embedding $\theta: C_0([1, \omega^{\tau}]) \to C_0([1, \omega^{\delta_i}])$. Then by lemma 2.6.7 (a), $\tau \leq \delta_i < \delta$, which is a contradiction.

For (b) suppose $C_0(X) \sim C_0(Y)$. There is $n \in \mathbb{N}$ such that $\operatorname{supp} Z_1 \subset Z_2 \oplus [1, \alpha\cdot\eta\cdot n]$ and ξ_i in the fixed sequence associated with ξ such that $\operatorname{supp}[1, \alpha\cdot\eta\cdot(n+1)] \subset Z_1 \oplus [1, \alpha\cdot\xi_i]$. By lemma 2.6.3 and the fact that $C_0([1, \alpha\cdot\eta\cdot(n+1)]) \sim C_0([1, \alpha\cdot\eta])$ (theorem 2.5.13 (c)), there is a linear embedding $\theta: C_0([1, \alpha\cdot\eta]) \to C_0([1, \alpha\cdot\xi_i])$, so by lemma 2.6.7 (b), $\overline{\eta} \leq \overline{\xi_i} < \overline{\xi}$, which is a contradiction. □

2.6.12 LEMMA: *Let α be an initial ordinal.*

(a) Let $X = Z_1 \oplus [1, \omega^{\delta})$ and $Y = Z_2 \oplus [1, \omega^{\tau})$, where Z_1 and Z_2 are compact spaces, δ, τ are σ-prime components or 1, $1 \leq \delta \leq \tau$, and $\overline{\omega^{\delta}} = \overline{\omega^{\tau}} = \overline{\alpha}$. If α is singular, $\alpha = \omega$ or $\omega^{\delta} \geq \alpha^2$, then $C_0(X) \sim C_0(Y)$ implies $\delta = \tau$.

(b) Let $X = Z_1 \oplus [1, \alpha\cdot\xi)$ and $Y = Z_2 \oplus [1, \alpha\cdot\eta)$, where Z_1 and Z_2 are compact spaces, ξ and η are σ-initial or of the form $\tau\cdot\omega$ with τ initial, and $\omega \leq \xi \leq \eta \leq \alpha$. If α is uncountable regular, then $C_0(X) \sim C_0(Y)$ implies $\xi = \eta$.

PROOF: For (a) suppose $\delta < \tau$. By lemma 2.6.8 (a), $\delta > 1$. By corollary 1.2.15 (b), there is τ_i in the fixed sequence associated with τ such that $\operatorname{supp} Z_1 \subset Z_2 \oplus [1, \omega^{\tau_i}]$. Now let $j > i$. Again by corollary 1.2.15 (b), there is δ_k in the fixed sequence associated with δ such that $\operatorname{supp}[1, \omega^{\tau_j}] \subset Z_1 \oplus [1, \omega^{\delta_k}]$. By lemma 2.6.3 there is a linear embedding $\theta: C_0([1, \omega^{\tau_j}]) \to C_0([1, \omega^{\delta_k}])$. So by lemma 2.6.7 (a), we have $\tau_j < \delta_k\cdot\omega \leq \delta\cdot\omega$, which implies $\delta < \tau \leq \delta\cdot\omega$. So since δ and τ are prime components, we have $\tau = \delta\cdot\omega$. But this contradicts lemma 2.6.11.

For (b) suppose $\xi < \eta$. There is $\eta_i \geq 1$ in the fixed sequence associated with η such that $\operatorname{supp} Z_1 \subset Z_2 \oplus [1, \alpha\cdot\eta_i]$. For $j > i$, there is $\xi_k \geq 1$ in the fixed sequence associated with ξ such that $\operatorname{supp}[1, \alpha\cdot\eta_j] \subset Z_2 \oplus [1, \alpha\cdot\xi_k]$. By lemma 2.6.3, there is a linear embedding $\theta: C_0([1, \alpha\cdot\eta_j]) \to C_0([1, \alpha\cdot\xi_k])$. By lemma 2.6.7 (b), $\overline{\eta_j} \leq \overline{\xi_k} \leq \overline{\xi}$. So $\overline{\eta} \leq \overline{\xi} \leq \overline{\eta}$ and hence $\overline{\eta} = \overline{\xi}$. Now we have four cases:

Case 1: ξ, η are initial.

Since $\xi < \eta$, we then have $\overline{\xi} < \overline{\eta}$. Contradiction.

Case 2: ξ is initial, $\eta = \tau\cdot\omega$ with τ initial.

Then $\overline{\xi} = \overline{\eta} = \overline{\tau}$, so $\tau = \xi$. But then we have a contradiction with lemma 2.6.11 (b).

Case 3: $\xi = \tau \cdot \omega$ with τ initial and η is initial.

By the same arguments as in case 3 we can derive a contradiction.

Case 4: $\xi = \tau \cdot \omega$ and $\eta = \delta \cdot \omega$ with τ and δ initial.

Then $\overline{\tau} = \overline{\delta}$ and so $\tau = \delta$, so $\xi = \eta$, which is a contradiction. □

2.6.13 LEMMA: *Let α be an uncountable regular ordinal, and $X = [1, \omega^\mu] \oplus [1, \beta)$, where μ is a prime component, $\beta \leq \omega^\mu$, β a σ-ordinal, $\omega^\mu > \alpha^2$ and $\overline{\omega^\mu} = \overline{\alpha}$. Let $Y = [1, \alpha \cdot \xi] \oplus [1, \gamma)$, where ξ is initial, $1 \leq \xi \leq \alpha$, γ a σ-ordinal and $\gamma \leq \alpha \cdot \xi$. Then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.*

PROOF: To the contrary suppose $C_0(X) \sim C_0(Y)$. There is $\delta < \gamma$ such that $\operatorname{supp}[1, \omega^\mu] \subset [1, \alpha \cdot \xi] \oplus [1, \delta] \approx [1, \alpha \cdot \xi]$. By corollary 2.6.4, there is a linear embedding $\theta: C_0([1, \omega^\mu]) \to C_0([1, \alpha \cdot \xi])$. But then by lemma 2.6.7 (a), $\omega^\mu \leq \alpha \cdot \xi \leq \alpha^2$, which contradicts the fact that $\omega^\mu > \alpha^2$. □

2.6.14 LEMMA: *Let α be an uncountable regular ordinal, and $X = Z_1 \oplus [1, \alpha \cdot \xi)$ and $Y = Z_2 \oplus [1, \omega^\mu)$, where Z_1 and Z_2 are compact spaces, $\xi \leq \alpha$ is σ-initial or of the form $\tau \cdot \omega$ with τ initial, μ is a σ-prime component, $\omega^\mu > \alpha^2$ and $\overline{\omega^\mu} = \overline{\alpha}$. Then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.*

PROOF: To the contrary suppose $C_0(X) \sim C_0(Y)$. By corollary 1.2.15 (b) there is μ_i in the fixed sequence associated with μ such that $\operatorname{supp} Z_1 \subset Z_2 \oplus [1, \omega^{\mu_i}]$. Let $j > i$ such that $\omega^{\mu_j} > \alpha^2$. By corollary 1.2.15 (b) there is $k \in \mathbb{N}$ such that $\operatorname{supp}[1, \omega^{\mu_j}] \subset Z_1 \oplus [1, \alpha \cdot \xi_k]$. Notice that $\xi_k < \xi \leq \alpha$. By lemma 2.6.3 there is a linear embedding from $C_0([1, \omega^{\mu_j}])$ into $C_0([1, \alpha \cdot \xi_k])$. Since $\alpha^2 < \omega^{\mu_j}$, there is also a linear embedding from $C_0([1, \alpha^2])$ into $C_0([1, \omega^{\mu_j}])$, thus there is a linear embedding $\theta: C_0([1, \alpha^2]) \to C_0([1, \alpha \cdot \xi_k])$. So by lemma 2.6.7 (b), $\overline{\alpha} \leq \overline{\xi_k}$ and hence $\alpha \leq \xi_k$. Contradiction. □

We now come to the announced

2.6.15 COROLLARY:

(a) *Let $X \in \mathcal{A}$ and $Y \in \mathcal{A}$. Then $C_p(X) \sim C_p(Y)$ if and only if $C_0(X) \sim C_0(Y)$ if and only if $X = Y$.*

(b) *Let $X \in \mathcal{B}$ and $Y \in \mathcal{B}$. Then $C_p(X) \sim C_p(Y)$ if and only if $C_0(X) \sim C_0(Y)$ if and only if $X = Y$.*

PROOF: If $C_p(X) \sim C_p(Y)$, then by remark 2.6.2, $C_0(X) \sim C_0(Y)$. Now suppose $C_0(X) \sim C_0(Y)$. Let α and β be ordinals such that $X \in \mathcal{A}^{(\alpha, \beta)}$. But then by lemma 2.6.9, $Y \in \mathcal{A}^{(\alpha, \beta)}$. By lemmas 2.6.10, 2.6.12, 2.6.13 and 2.6.14 it then follows that $X = Y$. If $X = Y$, then evidently $C_p(X) \sim C_p(Y)$ and (a) is proved.

Similarly (b) follows from lemmas 2.6.9, 2,6,12 and 2.6.14.

2.6.16 LEMMA: *Let $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Then $C_0(X)$ is not linearly homeomorphic to $C_0(Y)$.*

PROOF: $X = [1, \phi] \oplus [1, \psi)$ and $Y = [1, \xi)$ with ϕ a prime component, ψ and ξ σ-ordinals and $\phi \geq \psi$. Suppose $C_0(X) \sim C_0(Y)$. By theorem 2.5.1 (a), $\overline{X} = \overline{Y}$ and therefore $\overline{\psi} \leq \overline{\xi}$. As in lemma 2.6.9 we can derive $\overline{\psi} = \overline{\xi}$. Let α be the initial ordinal such that $\overline{\psi} = \overline{\xi} = \overline{\alpha}$. By lemma 2.6.14 we have to consider two cases:

Case 1: α is singular, $\alpha = \omega$ or ψ, $\xi > \alpha^2$.

Then $\psi = \omega^\tau$ and $\xi = \omega^\mu$ with μ and τ σ-prime components or 1. By lemma 2.6.12 (a), $\mu = \tau$ and by lemma 2.6.8 (b) $\tau > 1$. There is $\tau_i < \tau$ such that $\operatorname{supp}[1, \phi] \subset [1, \omega^{\tau_i}]$. So there is a linear embedding $\theta: C_0([1, \phi]) \to C_0([1, \omega^{\tau_i}])$ (corollary 2.6.4). By lemma 2.6.7, $\phi < \omega^{\tau_i \cdot \omega} \leq \omega^\tau = \psi$. Contradiction.

Case 2: α is uncountable and regular.

Then $\psi = \alpha \cdot \eta$ and $\xi = \alpha \cdot \tau$ with η and τ σ-initial or of the form initial$\cdot\omega$, $\omega \leq \eta$, $\tau \leq \alpha$. By lemma 2.6.12 (b), $\eta = \tau$. There is $i \in \mathbb{N}$ such that $\operatorname{supp}[1, \phi] \subset [1, \alpha \cdot \eta_i]$. So by corollary 2.6.4, there is a linear embedding $\theta: C_0([1, \phi]) \to C_0([1, \alpha \cdot \eta_i])$. Since $\alpha \cdot \eta_i < \alpha \cdot \eta = \psi \leq \phi$, there is a linear embedding $\theta': C_0([1, \alpha \cdot \eta_i]) \to C_0([1, \phi])$. This means that $C_0([1, \alpha \cdot \eta_i])$ and $C_0([1, \phi])$ have the same linear dimension. So by theorem 2.5.1 (c) $\phi = \alpha \cdot \gamma + \delta$ for some $\gamma \leq \alpha$ and $\delta < \alpha$ with $\overline{\gamma} = \overline{\eta_i}$. But then $\overline{\gamma} < \overline{\eta}$, so $\gamma < \eta$, which implies $\phi < \alpha \cdot \eta = \psi$. Contradiction. □

The following theorem gives the classification announced in the introduction of this section.

2.6.17 THEOREM: *Let α and β be σ-ordinals Then the following statements are equivalent:*

(1) $C_p([1, \alpha)) \sim C_p([1, \beta))$

(2) $C_0([1, \alpha)) \sim C_0([1, \beta))$

(3) *There are compacta X_i and Y_i ($i \in \mathbb{N}$) such that* $[1, \alpha) = \oplus_{i=1}^{\infty} X_i$, $[1, \beta) = \oplus_{i=1}^{\infty} Y_i$ *and for every* $i \in \mathbb{N}$, $C_p(X_i) \sim C_p(Y_i)$.

(4) *There are compacta* X_i *and* Y_i $(i \in \mathbb{N})$ *such that* $[1, \alpha) = \oplus_{i=1}^{\infty} X_i$, $[1, \beta) = \oplus_{i=1}^{\infty} Y_i$ *and for every* $i \in \mathbb{N}$, $C_0(X_i) \sim C_0(Y_i)$.

(In fact the X_i *and the* Y_i *are compact ordinal spaces.)*

PROOF: For (1) $\Rightarrow$ (2) apply corollary 1.2.21. Furthermore (2) $\Rightarrow$ (4) easily follows from lemma 2.6.6, corollary 2.6.15 and lemma 2.6.16. For (4) $\Rightarrow$ (3) notice that for compact ordinals we have the same isomorphical classification for the topology of pointwise convergence and the compact-open topology (section 2.5), so $C_0(X_i) \sim C_0(Y_i)$ implies $C_p(X_i) \sim C_p(Y_i)$. Finally (3) $\Rightarrow$ (1) follows from lemma 2.3.7, and the theorem is proved. □

2.6.18 EXAMPLE: Notice that $[1, \omega^\omega) = \oplus_{i=1}^{\infty} [1, \omega^n]$ and $[1, \omega^2) = \oplus_{i=1}^{\infty} [1, \omega]$. By theorem 2.4.7, $C_p([1, \omega^n]) \sim C_p([1, \omega])$ for each $n \in \mathbb{N}$ (because $\omega \leq \omega^n < \omega^\omega$). So by theorem 2.6.17, $C_p([1, \omega^2)) \sim C_p([1, \omega^\omega))$.

With the next lemma and theorems 2.6.17, 2.4.1 and 2.4.7, we have obtained a complete isomorphical classification for the spaces $C_p(X)$ and $C_0(X)$ for σ-compact ordinal spaces X. Notice that from the classification it follows that for these spaces $C_p(X) \sim C_p(Y)$ if and only if $C_0(X) \sim C_0(Y)$.

2.6.19 LEMMA: *Let* α *and* β *be ordinals such that* $C_0([1, \alpha)) \sim C_0([1, \beta))$. *Then*

(*a*) α *is a successor if and only if* β *is a successor, and*

(*b*) α *is a* σ*-ordinal if and only if* β *is a* σ*-ordinal.*

PROOF: For (a), if α is a successor, then $[1, \alpha)$ is compact. So by theorem 1.5.7 $[1, \beta)$ is compact and thus β is a successor.

For (b), let α be a σ-ordinal. By remark 2.6.2 $[1, \alpha)$ is a non-compact non-pseudocompact space, so by theorem 1.5.7 $[1, \beta)$ is a non-compact non-pseudocompact space. But then by remark 2.6.2, β is a σ-ordinal. □

By the obtained classification theorems we conclude that for locally compact spaces X and Y and their respective one-point compactifications ωX and ωY, the fact that $C_p(X) \sim C_p(Y)$ does not necessarily imply that $C_p(\omega X) \sim C_p(\omega Y)$, and vica versa. For example, $C_p([1, \omega^\omega))$ is linearly homeomorphic to $C_p([1, \omega^2))$ (example 2.6.18), however $C_p([1, \omega^\omega])$ is not linearly homeomorphic to $C_p([1, \omega^2])$. Furthermore, $C_p([1, \omega])$ is linearly homeomorphic to $C_p([1, \omega^2])$, but $C_p([1, \omega))$ is not linearly homeomorphic to $C_p([1, \omega^2))$ (lemma 2.6.8).

The same remark applies to the compact-open topology.

The question now arises whether we can derive a similar classification for the spaces of bounded continuous functions. This seems impossible by the methods of this section. Simply observe that corollary 1.2.15 (b) plays a fundamental role, and that it does not hold for spaces of bounded continuous functions (example 1.2.12). In section 4.6 we will come back to this and we will show there that for σ-ordinals α, theorem 2.6.17 does not hold for the spaces $C_p^*([1, \alpha))$.

Another question is whether a similar classification can be derived for arbitrary ordinal spaces. Again it seems that this is impossible by the methods of this section, because we essentially used that every closed and bounded subset of $[1, \alpha)$ is compact (with α a σ-ordinal), and by remark 2.6.2 this is not true for the spaces $[1, \alpha)$ if α is an ordinal with $cf(\alpha) > \omega$.

Finally we remark that the results in this section are new. They are extensions of the results in [3] for the countable case.

§2.7. Separable metric zero-dimensional locally compact spaces

In this section we will give a complete isomorphical classification of the function spaces $C_p(X)$ and $C_0(X)$ with X a separable metric zero-dimensional locally compact space. Notice that for separable metric zero-dimensional compact spaces X we already have a complete classification of the spaces $C_0(X)$ (cf. theorem 2.4.1) and $C_p(X)$ (cf. theorem 2.4.7). This classification is such that for two spaces X and Y it follows that $C_p(X)$ is linearly homeomorphic to $C_p(Y)$ if and only if $C_0(X)$ is linearly homeomorphic to $C_0(Y)$ (cf. remark 2.4.8). By theorems 1.5.1 and 1.5.4 it remains to present a complete classification of the spaces $C_0(X)$ and $C_p(X)$ with X separable metric zero-dimensional locally compact but not compact. For convenience *in this section every space is separable metric*.

2.7.1 LEMMA: *Let X be a countable space which is locally compact but not compact. Then there is a σ-limit ordinal α such that $X \approx [1, \alpha)$.*

PROOF: Let ωX be the Alexandroff one-point compactification of X. By proposition

2.2.7 and theorem 2.2.8, there is a limit ordinal λ such that $\omega X \approx [1, \lambda]$. So X is a dense subset of $[1, \lambda]$ such that $[1, \lambda] \setminus X$ contains only one point, say μ. Since X is dense in $[1, \lambda]$, μ is a limit ordinal. So

$$X = [1, \lambda] \setminus \{\mu\} = [1, \mu) \oplus [\mu + 1, \lambda] = [\mu + 1, \lambda] \oplus [1, \mu) = [\mu + 1, \lambda + \mu) \approx [1, \alpha)$$

for some limit ordinal α. Since α is countable, α is a σ-limit ordinal. □

By lemma 2.7.1 countable spaces which are locally compact but not compact are homeomorphic to ordinal spaces. Since these ordinals are σ-ordinals we already have a complete classification for their function spaces $C_p(X)$ and $C_0(X)$ (cf. theorem 2.6.17). We shall now consider the case of uncountable locally compact spaces which are not compact. The proof of their classification is similar to the one in section 2.6. We define a class of spaces such that for every uncountable zero-dimensional space X which is locally compact but not compact, $C_p(X)$ is linearly homeomorphic to a space in this class. After that, we prove that two different spaces in this class are not linearly homeomorphic, which gives the classification.

2.7.2 LEMMA: *Let X be an uncountable zero-dimensional space which is locally compact but not compact. Then there is a decomposition $\bigoplus_{i=1}^{\infty} X_i$ of X consisting of compacta such that either every X_i is uncountable or X_i is uncountable iff $i = 1$.*

PROOF: Let $X = \bigoplus_{i=1}^{\infty} Z_i$ be a decomposition of X consisting of compacta (this is possible because X is zero-dimensional).

Case 1: Only finitely many Z_i are uncountable.

Let $n = \max\{ i : Z_i \text{ is uncountable}\}$. Let $X_1 = Z_1 \oplus \cdots \oplus Z_n$ and $X_i = Z_{n+i-1}$ $(i \geq 2)$.

Case 2: Infinitely many Z_i are uncountable.

Suppose $Z_{i_1}, Z_{i_2}, \ldots$ are uncountable. Let $X_n = Z_{i_{n-1}+1} \oplus \cdots \oplus Z_{i_n}$ $(i_0 = 0)$. Since X_n is compact and uncountable we are done. □

We now define the class $\mathscr{C} \cup \mathscr{D}$ of spaces as follows:

$$\mathscr{C} = \{C \oplus [1, \omega^\tau) : C \text{ is the Cantor set and } 1 \leq \tau < \omega_1 \text{ is a prime component}\},$$
$$\mathscr{D} = \{\bigoplus_{i=1}^{\infty} C_i : C_i \text{ is a copy of the Cantor set}\}.$$

Observe the following:

If $X \in \mathscr{C}$, say $X = C \oplus [1, \omega^\tau)$, then $X = C \oplus [1, \omega^{\tau_1}] \oplus [1, \omega^{\tau_2}] \oplus \ldots$ where $(\tau_i)_i$ is the fixed sequence cofinal with τ which was chosen on page 90. If for $X \in \mathscr{C}$ we write $X = \bigoplus_{i=1}^{\infty} X_i$, then we implicitly mean that the X_i are as above. If $X \in \mathscr{D}$ then we consid-

er the fixed decomposition $\oplus_{i=1}^{\infty} C_i$.

2.7.3 LEMMA: *Let X be an uncountable zero-dimensional space which is locally compact but not compact. Then there is a decomposition $\oplus_{i=1}^{\infty} X_i$ of X and a space $Y \in \mathcal{C} \cup \mathcal{D}$ such that $C_p(X_i) \sim C_p(Y_i)$ (where Y_i is the i-th component of the decomposition of Y stated as above). In particular $C_p(X) \sim C_p(Y)$ and $C_0(X) \sim C_0(Y)$.*

PROOF: By lemma 2.7.2 there is a decomposition $\oplus_{i=1}^{\infty} X_i'$ of X consisting of compacta, such that either every X_i' is uncountable or X_i' is uncountable iff $i = 1$.
Case 1: X_i' is uncountable iff $i = 1$.

Since $X' = \oplus_{i=2}^{\infty} X_i'$ is a countable space which is locally compact but not compact, by lemma 2.7.1 and lemma 2.6.6 there is a decomposition $\oplus_{i=1}^{\infty} Z_i$ of X' and a space $Y' \in \mathcal{A} \cup \mathcal{B}$ such that $C_p(Z_i) \sim C_p(Y_i')$. By lemma 2.6.9 $Y' \in \mathcal{A}^{(\omega,\omega)} \cup \mathcal{B}^{(\omega,\omega)}$, because X' is countable.

If $Y' \in \mathcal{A}^{(\omega,\omega)}$, then $Y' = [1, \omega^{\mu}] \oplus [1, \omega^{\tau})$, where μ and τ are prime components such that $1 \leq \mu, \tau < \omega_1$. Then $C_p(Z_1) \sim C_p([1, \omega^{\mu}])$. Let $X_1 = X_1' \oplus Z_1$ and for $i \geq 2$ let $X_i = Z_i$. Since X_1 is zero-dimensional, uncountable and compact, by theorem 2.4.7 $C_p(X_1) \sim C_p(C)$. So if we let $Y = C \oplus [1, \omega^{\tau})$ we are done.

If $Y' \in \mathcal{B}^{(\omega,\omega)}$, say $Y' = [1, \omega^{\tau})$ with τ a prime component, $1 \leq \tau < \omega_1$, then let $Y = C \oplus [1, \omega^{\tau})$, $X_1 = X_1'$ and for $i \geq 2$, $X_i = Z_i$.
Case 2: Every X_i' is uncountable.

Define $Y = \oplus_{i=1}^{\infty} C_i$. By theorem 2.4.7 $C_p(X_i) \sim C_p(C_i)$, so let $X_i = X_i'$. □

2.7.4 LEMMA:

(a) *If $X, Y \in \mathcal{C}$, then $C_p(X) \sim C_p(Y)$ if and only if $C_0(X) \sim C_0(Y)$ if and only if $X = Y$.*

(b) *If $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, then $C_0(X)$ and $C_0(Y)$ are not linearly homeomorphic.*

PROOF: Part (a) follows directly from lemma 2.6.12 (a).

For (b), suppose that $X = C \oplus [1, \omega^{\tau})$ and $Y = \oplus_{i=1}^{\infty} C_i$. Assume $C_0(X) \sim C_0(Y)$. There is $n \in \mathbb{N}$ such that $\operatorname{supp} C \subset C_1 \oplus \cdots \oplus C_n$ (corollary 1.2.15 (b)). There is $i \in \mathbb{N}$ such that $\operatorname{supp} C_{n+1} \subset C \oplus [1, \omega^{\tau_i}]$. So by lemma 2.6.3, there is an embedding $\phi: C_0(C) \to C_0([1, \omega^{\tau_i}])$. Since by theorem 2.4.1 (c) we have $C_0(C) \sim C_0(C \oplus [1, \omega^{\tau \cdot \omega}])$, we have a linear embedding $\theta: C_0([1, \omega^{\tau \cdot \omega}]) \to C_0([1, \omega^{\tau_i}])$. But then by lemma 2.6.7 (a), $\tau \cdot \omega < \tau_i \cdot \omega$. This is a contradiction. □

2.7.5 THEOREM: *Let X and Y be uncountable zero-dimensional spaces which are both locally compact but not compact. Then the following statements are equivalent:*

(1) $C_p(X) \sim C_p(Y)$

(2) $C_0(X) \sim C_0(Y)$

(3) *There are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \oplus_{i=1}^{\infty} X_i$, $Y = \oplus_{i=1}^{\infty} Y_i$ and $C_p(X_i) \sim C_p(Y_i)$.*

(4) *There are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \oplus_{i=1}^{\infty} X_i$, $Y = \oplus_{i=1}^{\infty} Y_i$ and $C_0(X_i) \sim C_0(Y_i)$.*

PROOF: For (1) $\Rightarrow$ (2) apply corollary 1.2.21. Furthermore (2) $\Rightarrow$ (4) follows easily from lemmas 2.7.3 and 2.7.4. For (4) $\Rightarrow$ (3) notice that for compact zero-dimensional spaces we have the same isomorphical classification for the topology of pointwise convergence and the compact-open topology (section 2.5), so $C_0(X_i) \sim C_0(Y_i)$ implies $C_p(X_i) \sim C_p(Y_i)$. Finally (3) $\Rightarrow$ (1) follows from lemma 2.3.7. □

REMARK: In view of the remark after theorem 2.6.17 we have the following: Let X and Y be spaces such as in theorem 2.7.5 and let ωX and ωY be their respective one point compactifications. By theorem 2.4.7, $C_p(\omega X)$ is linearly homeomorphic to $C_p(\omega Y)$, irrespective of whether $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.

Again, the same remark applies to the compact-open topology.

We almost completed the isomorphical classification of the function spaces $C_p(X)$ and $C_0(X)$ of locally compact zero-dimensional spaces X. It remains to distinguish between "countable" and "uncountable". For the pointwise topology, if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, we have that X is countable if and only if Y is countable (by theorem 1.5.9). The same holds for the compact open topology as is shown by the following

2.7.6 PROPOSITION: *Let X and Y be locally compact zero-dimensional spaces such that $C_0(X)$ and $C_0(Y)$ are linearly homeomorphic. Then X is countable if and only if Y is countable.*

PROOF: Suppose that X is countable and Y is uncountable. By theorem 1.5.4b and theorem 2.4.1 we may assume that X and Y are not compact. By lemma 2.6.6 and lemma 2.7.3 we may assume that $X \in \mathcal{A} \cup \mathcal{B}$ and $Y \in \mathcal{C} \cup \mathcal{D}$. There is a clopen copy of C in Y. Then $\operatorname{supp} C$ is contained in a clopen copy of $[1, \alpha]$ in X for some countable ordinal α. So by corollary 2.6.4 there is a linear embedding from $C_0(C)$ into $C_0([1, \alpha])$. Since

$C_0(C \oplus [1, \alpha^\omega]) \sim C_0(C)$ (theorem 2.4.1) we then have a linear embedding $\theta: C_0([1, \alpha^\omega]) \to C_0([1, \alpha])$. But this is impossible by corollary 2.5.3. □

Notice that by lemma 1.4.1, proposition 1.4.3, theorems 1.5.1, 1.5.4, 2.4.1, 2.4.7, 2.6.17, 2.7.5 and proposition 2.7.6 we have as announced in the introduction of this chapter that for locally compact zero-dimensional spaces X and Y, $C_p(X)$ is linearly homeomorphic to $C_p(Y)$ if and only if $C_0(X)$ is linearly homeomorphic to $C_0(Y)$.

Finally we remark that the main results of this section were published in [3].

CHAPTER 3

On topological equivalence of function spaces

All spaces considered in this chapter are separable and metrizable.

Let h be a homeomorphism between $\mathbb{R}$ and $(-1, 1)$. Then for each space X and each bounded $f: X \to \mathbb{R}$, there is $m \in \mathbb{N}$ such that $(h \circ f)(X) \subset [-1+1/m, 1-1/m]$. This allows us to identify $C_p(X)$ and the subspace

$$\{f : X \to (-1, 1) : f \text{ is continuous}\}$$

of $(-1, 1)^X$; similarly we can identify $C_p^*(X)$ and

$$\{f \in C_p(X) : \text{there is } m \in \mathbb{N} \text{ such that } f(X) \subset [-1+\frac{1}{m}, 1-\frac{1}{m}]\}.$$

In particular if X is countable, $C_p(X)$ and $C_p^*(X)$ can be regarded as subspaces of the Hilbert cube.

Let $X = \{x_0, x_1, x_2, \ldots\}$ be a countable space, $C_{p,0}(X) = \{f \in C_p(X) : f(x_0) = 0\}$ and $C_{p,0}^*(X) = \{f \in C_p^*(X) : f(x_0) = 0\}$. In this chapter we mainly consider non-locally compact countable spaces. For $X = \{x_0, x_1, x_2, \ldots\}$ not locally compact, we assume that X is not locally compact at x_0.

In [38], van Mill showed that for a non-locally compact countable space X, $C_p^*(X)$ is homeomorphic to σ_ω, where

$$\sigma_\omega = (\ell_f^2)^\infty \text{ and } \ell_f^2 = \{x \in \ell^2 : x_i = 0 \text{ for all but finitely many } i\}$$

(ℓ^2 denotes separable Hilbert space).

One of our main results in this chapter is that for a non-locally compact countable space X, $C_p(X)$ is homeomorphic to σ_ω. We will give two proofs. The first proof in section 3.2 is quite technical: Among other things we prove that whenever Y is any other non-locally compact countable space, then there exists a homeomorphism from the Hilbert cube onto itself arbitrary close to the identity which maps $C_{p,0}(X)$ onto $C_{p,0}(Y)$. The second proof in section 3.3 is in the spirit of van Mill's proof that $C_p^*(X)$

and σ_ω are homeomorphic [38]. The strategy followed depends strongly on results of Toruńczyk [50], [51]. It is less technical than the proof in section 3.2, but it gives only that $C_p(X)$ and σ_ω are homeomorphic.

Section 3.1 contains preliminaries from infinite-dimensional topology which will be used to prove our main results. In that section we also present some results on Q-matrices

The question remains whether for non-locally compact countable spaces X and Y, there is a homeomorphism from the Hilbert cube onto itself arbitrary close to the identity which maps $C^*_{p,0}(X)$ onto $C^*_{p,0}(Y)$. In section 3.2 we give a positive answer to this question.

In section 3.4 we give some final remarks. We state recent theorems of Dobrowolski, Gulko and Mogilski [20] and Cauty [16] from which can be concluded that for a non-discrete countable space X, $C_p(X)$ and $C^*_p(X)$ are homeomorphic to σ_ω. Since for any countable discrete space X, $C_p(X)$ is homeomorphic to $\mathbb{R}^n$, where $n \in \mathbb{N} \cup \{\infty\}$ is the cardinality of X, we obtain a complete topological classification of the spaces $C_p(X)$, for countable spaces X. Furthermore in that section we state the uniform classification derived by Gulko [27] of the uniform spaces $C_p(X)$, for countable infinite compact spaces X.

§3.1. Preliminaries and Q-matrices

In this chapter we consider products of spaces at several places. It will be convenient to explicitly define an admissible metric on such a product. For every $i \in \mathbb{N}$, let P_i be a space with an admissible metric d_i such that each d_i is bounded by c for a fixed $c \in \mathbb{R}$. If we have a finite product of spaces $P = \Pi_{i=1}^n P_i$ then the specific admissible metric d on P is defined by $\mathrm{d} = \max\{\mathrm{d}_1, \ldots, \mathrm{d}_n\}$, and if we have a countable infinite product of spaces $P = \Pi_{i=1}^\infty P_i$ then the specific admissible metric d on P is defined by

$$\mathrm{d}(x, y) = \sum_{i=1}^{\infty} 2^{-i} \mathrm{d}_i(x_i, y_i),$$

where $x = (x_i)_{i \in \mathbb{N}}$, $y = (y_i)_{i \in \mathbb{N}} \in P$. Whenever for each $i \in \mathbb{N}$, $P_i = X$ for some space X, we denote P by X^∞.

Consider ***the Hilbert cube*** $Q = \Pi_{i=1}^\infty [-1, 1]_i$, where $[-1, 1]_i = [-1, 1]$ for every $i \in \mathbb{N}$. Then the topology of Q is given by the metric

$$\mathrm{d}(x, y) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|,$$

where $x=(x_i)_{i\in\mathbb{N}}$, $y=(y_i)_{i\in\mathbb{N}}\in Q$. The subset $s=\prod_{i=1}^{\infty}(-1,1)_i$ of Q, where $(-1,1)_i=(-1,1)$ for every $i\in\mathbb{N}$, is called ***the pseudo-interior*** of Q. $B(Q)=Q\setminus s$ is called ***the pseudo-boundary*** of Q. A space which is homeomorphic to Q is called ***a Hilbert cube***.

A subspace A of separable Hilbert space ℓ^2 will be called a ***Keller space*** whenever it is compact, convex and infinite dimensional. In [33] it is proved that a Keller space is a Hilbert cube (see also [39]). Since there is an affine embedding from Q into ℓ^2, we obtain the following

3.1.1 THEOREM: *A Keller space in Q is a Hilbert cube.*

For spaces X and Y let

$$C(X,Y)=\{f: X\to Y : f \text{ is continuous}\}$$

and

$$\mathcal{H}(X,Y)=\{f: X\to Y : f \text{ is a homeomorphism}\}.$$

Whenever $X=Y$ we write $\mathcal{H}(X)$ for $\mathcal{H}(X,X)$. For $f, g\in C(X,Y)$ we define

$$\hat{d}(f,g)=\sup\{d(f(x),g(x)) : x\in X\}\in[0,\infty],$$

where d is an admissible metric on Y. As is easily seen we have the following

3.1.2 LEMMA: *Let X, Y and Z be spaces. Let* $f, g\in C(Y,Z)$ *and* $h\in C(X,Y)$. *Then* $\hat{d}(f\circ h, g\circ h)\le\hat{d}(f,g)$. *If moreover h is surjective, then* $\hat{d}(f\circ h, g\circ h)=\hat{d}(f,g)$.

Let X be a compact space and let A be a closed subspace of X. Then A is a ***Z-set*** in X if and only if for every $f\in C(Q,X)$ and for every $\varepsilon>0$, there is a $g\in C(Q,X)$ such that

(a) $\hat{d}(f,g)<\varepsilon$, and
(b) $g(Q)\cap A=\emptyset$.

The definition of a Z-set is independent from the chosen metric on X. By $\mathcal{Z}(X)$ we denote the family of all Z-sets in X. A countable union of Z-sets is called a ***σZ-set***. The family of all σZ-sets in X is denoted by $\mathcal{Z}_\sigma(X)$. An embedding $f: X\to Y$, where Y is another compact space, is called a ***$\mathcal{Z}$-embedding*** whenever $f(X)\in\mathcal{Z}(Y)$.

3.1.3 LEMMA ([39, Lemma 6.2.2]): *Let X be a space. Then*

(a) If $A\in\mathcal{Z}(X)$ *and* $B\subset A$ *is closed, then* $B\in\mathcal{Z}(X)$.

(b) If $A \in \mathcal{I}(X)$, then A has empty interior in X.

PROOF: (a) follows directly from the definition of a Z-set. For (b) suppose that $\operatorname{Int} A \neq \emptyset$. Let $x \in \operatorname{Int} A$ and put $\varepsilon = d(x, X \setminus \operatorname{Int} A)$. Let $f \in C(Q, X)$ be the constant function with value x. If $g \in C(Q, X)$ satisfies $\hat{d}(f, g) < \varepsilon/2$, then obviously $g(Q) \cap A \neq \emptyset$. Hence A is not a Z-set in X. □

3.1.4 LEMMA ([39, Lemma 6.2.3]): *Let $P = \prod_{i=1}^{\infty} P_i$ be a countable infinite product of compact spaces. Let $A \subset P$ be closed such that $\pi_j(A) \neq P_j$ for infinitely many j. Then $A \in \mathcal{I}(P)$.*

PROOF: Let $f = (f_1, f_2, \ldots) \in C(Q, P)$ and $\varepsilon > 0$. For each $i \in \mathbb{N}$, let d_i be an admissible metric on P_i bounded by 1. Find $j \in \mathbb{N}$ such that $2^{-j} < \varepsilon$. By assumption there are $k > j$ and $t \in P_k \setminus \pi_k(A)$. Define $g \in C(Q, P)$ by

$$g(x) = (f_1(x), \ldots, f_{k-1}(x), t, f_{k+1}(x), \ldots)$$

Then $g(Q) \cap A = \emptyset$ and

$$\begin{aligned} \hat{d}(f, g) &= \sup \{d(f(q), g(q)) : q \in Q\} \\ &= \sup \{\Sigma_{i=1}^{\infty} 2^{-i} d_i(f_i(q), g_i(q)) : q \in Q\} \\ &= \sup \{2^{-k} d_k(f_k(q), t) : q \in Q\} \\ &\leq 2^{-k} < 2^{-j} < \varepsilon. \end{aligned}$$ □

3.1.5 THEOREM ([39, Th. 6.4.6]): *Let $E, F \in \mathcal{I}(Q)$ and let $f : E \to F$ be a homeomorphism such that $\hat{d}(f, 1_E) < \varepsilon$. Then f can be extended to a homeomorphism $\tilde{f} : Q \to Q$ such that $\hat{d}(\tilde{f}, 1) < \varepsilon$.*

3.1.6 THEOREM ([39, Th. 6.4.8]): *Let X be a compact space, let $A \subset X$ be closed and let $f : X \to Q$ be continuous such that $f \mid A$ is a $\mathcal{I}$-embedding. Then for every $\varepsilon > 0$ there is a $\mathcal{I}$-embedding $g : X \to Q$ such that $\hat{d}(f, g) < \varepsilon$ and $g \mid A = f \mid A$.*

Let $\{A_n\}_{n \in \mathbb{N}}$ be an increasing family of Z-sets in a compact space X. Then $\{A_n\}_{n \in \mathbb{N}}$ is a ***skeleton*** in X whenever for every $\varepsilon > 0$, $n \in \mathbb{N}$ and $Z \in \mathcal{I}(X)$, there are $h \in \mathcal{H}(X)$ and $m \in \mathbb{N}$ such that

(a) $\hat{d}(h, 1) < \varepsilon$,
(b) $h \mid A_n = 1$, and
(c) $h(Z) \subset A_m$.

The definition of a skeleton is independent from the chosen metric on X. A subset A of X is called a ***skeletoid*** in X if there is a skeleton $\{A_n\}_{n\in\mathbb{N}}$ in X such that $A=\bigcup_{n=1}^{\infty}A_n$. Note that if A is a skeletoid in X and $h\in\mathcal{H}(X, Y)$, then $h(A)$ is a skeletoid in Y. In the following theorem sufficient conditions are given for an increasing sequence of Z-sets in a Keller space to be a skeleton.

3.1.7 THEOREM: *If $\{A_i\}_{i\in\mathbb{N}}$ is an increasing family of Z-sets in a Keller space P such that,*

(a) for every $i\in\mathbb{N}$, $A_i\in\mathcal{I}(A_{i+1})$,

(b) for every $i\in\mathbb{N}$, A_i is convex and infinite-dimensional, and

(c) $\bigcup_{i=1}^{\infty}A_i$ is dense in P,

then $\{A_i\}_{i\in\mathbb{N}}$ is a skeleton in P.

PROOF: Let $Z\in\mathcal{I}(P)$, $n\in\mathbb{N}$ and $\varepsilon>0$. Since P is a Hilbert cube, there is by theorem 3.1.5, $\delta>0$ such that if $E, F\in\mathcal{I}(P)$ and if $f: E\to F$ is a homeomorphism with $\hat{d}(f, 1_E)<\delta$, then f can be extended to a homeomorphism $\tilde{f}: P\to P$ such that $\hat{d}(\tilde{f}, 1)<\varepsilon$ (we use that a homeomorphism between P and Q is uniformly continuous).

Find $\{x_1, \ldots, x_k\}\subset P$ such that $P=\bigcup_{j=1}^{k}B(x_j, \delta/4)$. There is $m\geq n$ such that for each $j\leq k$, $B(x_j, \delta/4)\cap A_m\neq\emptyset$. By [39, Cor. 8.2.2] there is a retraction $r: P\to A_m$ such that for each $x\in P$, $d(x, r(x))=d(x, A_m)$. We claim that $\hat{d}(r, 1)<\delta/2$. Indeed let $x\in P$. Let $j\leq k$ be such that $x\in B(x_j, \delta/4)$. Find $y\in B(x_j, \delta/4)\cap A_m$. Then

$$d(x, r(x))=d(x, A_m)\leq d(x, y)\leq d(x, x_j)+d(x_j, y)<\frac{\delta}{2}.$$

Let $r'=r\mid(Z\cup A_n): Z\cup A_n\to A_m$. Then $r'\mid A_n$ is a $\mathcal{I}$-embedding. Note that A_m is a Keller space, hence a Hilbert cube. So by theorem 3.1.6 there is a $\mathcal{I}$-embedding $s: Z\cup A_n\to A_m$ such that $s\mid A_n=r'\mid A_n=1_{A_n}$ and $\hat{d}(s, r')<\delta/2$. Hence $\hat{d}(s, 1)<\delta$. Note that $Z\cup A_n\in\mathcal{I}(P)$ and $s(Z\cup A_n)\in\mathcal{I}(P)$ and $s: Z\cup A_n\to s(Z\cup A_n)$ is a homeomorphism. Hence there is $h\in\mathcal{H}(P)$ with $\hat{d}(h, 1)<\varepsilon$ and $h\mid Z\cup A_n=s$. This implies that $h(Z)\subset A_m$ and $h\mid A_n=1$. □

3.1.8 EXAMPLE: For every $n\in\mathbb{N}$, let $\Sigma_n=[-1+1/n, 1-1/n]^{\infty}$, and let $\Sigma=\bigcup_{n=1}^{\infty}\Sigma_n$. By lemma 3.1.4 we have for $i\in\mathbb{N}$, $\Sigma_i\in\mathcal{I}(Q)$ and $\Sigma_i\in\mathcal{I}(\Sigma_{i+1})$. Since each Σ_i is convex and infinite-dimensional and Σ is dense in Q it follows from theorem 3.1.7 that $\{\Sigma_n\}_{n\in\mathbb{N}}$ is a skeleton in Q, so that Σ is a skeletoid in Q.

Another well-known example of a skeletoid in Q is $B(Q)$.

There are interesting theorems on skeletoids. We mention a few which we will use in the sequel.

3.1.9 THEOREM ([17, Lemma 4.3]): *Let A and B be skeletoids in a Hilbert cube P. Let $Z \in \mathcal{I}(P)$ such that $Z \cap (A \cup B) = \emptyset$. Then for every $\varepsilon > 0$, there is $h \in \mathcal{H}(P)$ such that*

(a) $h(A) = B$,
(b) $h \mid Z = 1$, *and*
(c) $\hat{d}(h, 1) < \varepsilon$.

3.1.10 THEOREM ([17, Th. 6.7; 39, Th. 6.5.3 (2)]): *Let A be a skeletoid in a Hilbert cube P, $B \in \mathcal{I}(P)$ and $C \in \mathcal{I}_\sigma(P)$. Then $A \setminus B$ and $A \cup C$ are skeletoids in P.*

3.1.11 COROLLARY: *Let A be a skeletoid in Q, $B \in \mathcal{I}(Q)$ and $C \in \mathcal{I}_\sigma(Q)$ such that $C \subset B$. Then for every $\varepsilon > 0$ there is $h \in \mathcal{H}(Q)$ such that*

(a) $\hat{d}(h, 1) < \varepsilon$, *and*
(b) $h(B) \cap A = h(C)$.

PROOF: By theorem 3.1.10, $(A \setminus B) \cup C$ is a skeletoid. By theorem 3.1.9 there is $h \in \mathcal{H}(Q)$ such that

(1) $\hat{d}(h, 1) < \varepsilon$, and
(2) $h((A \setminus B) \cup C) = A$.

Then we have

$$\begin{aligned} h(B) \cap A &= h(B) \cap h((A \setminus B) \cup C) \\ &= h(B \cap ((A \setminus B) \cup C)) \\ &= h(B \cap C) \\ &= h(C). \end{aligned}$$ □

We now present the notions of a $\mathcal{I}$-matrix and a Q-matrix. These notions were introduced by van Mill in [38].

A ***$\mathcal{I}$-matrix*** in a compact space X is a collection $\mathcal{A} = \{ A^n_m : n, m \in \mathbb{N} \}$ of Z-sets in X such that for every $m, n \in \mathbb{N}$,

(a) $A^n_1 = \emptyset$,
(b) $A^n_m \subset A^n_{m+1}$, and
(c) $A^{n+1}_m \subset A^n_m$.

Define the ***kernel of*** $\mathcal{A}$ by $\ker\mathcal{A} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$. Then clearly $\ker\mathcal{A}$ is an $F_{\sigma\delta}$-subset of X.

Let $\mathcal{A} = \{ A_m^n : n, m \in \mathbb{N} \}$ be a $\mathcal{T}$-matrix in a compact space X. Then $\mathcal{A}$ is a ***Q-matrix*** if and only if $\mathcal{A}$ has the following properties:

(a) For every $n \in \mathbb{N}$, $\{ A_m^n \}_{m>1}$ is a skeleton in X,

and for every $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$,

(b) $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Hilbert cube,

(c) for every $p \in \mathbb{N}$, $\{ \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$, and

(d) for every $s, t \in \mathbb{N}$ such that $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subset A_t^s$ we have
$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathcal{T}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$.

Note that if $\mathcal{A} = \{ A_m^n : n, m \in \mathbb{N} \}$ is a Q-matrix in X and $h \in \mathcal{H}(X, Y)$, then $h(\mathcal{A}) = \{ h(A_m^n) : n, m \in \mathbb{N} \}$ is a Q-matrix in Y (we use that h is uniformly continuous).

Let $\mathcal{A} = \{ A_m^n : n, m \in \mathbb{N} \}$ be a $\mathcal{T}$-matrix and let $A_{m_1}^{n_1}$ and $A_{m_2}^{n_2}$ be in $\mathcal{A}$ such that $n_1 < n_2$ and $m_1 \geq m_2$. Then $A_{m_2}^{n_2} \subset A_{m_1}^{n_1}$ so $A_{m_1}^{n_1} \cap A_{m_2}^{n_2} = A_{m_2}^{n_2}$. So for $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$ we may assume $i_1 < \cdots < i_m$ if we are interested in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$.

3.1.12 THEOREM ([38]): *If $\mathcal{A}$ and $\mathcal{B}$ are Q-matrices in Q, then*

(a) $\ker\mathcal{A}$ *is homeomorphic to* σ_ω, *and*

(b) for every $\varepsilon > 0$ *there is* $h \in \mathcal{H}(Q)$ *such that* $\hat{d}(h, 1) < \varepsilon$ *and* $h(\ker\mathcal{A}) = \ker\mathcal{B}$.

3.1.13 COROLLARY: *Let P_1 and P_2 be Hilbert cubes and let $\mathcal{A}$ and $\mathcal{B}$ be Q-matrices in P_1 resp. P_2. Then*

(a) $\ker\mathcal{A}$ *is homeomorphic to* σ_ω, *and*

(b) for every $h \in \mathcal{H}(P_1, P_2)$ *and* $\varepsilon > 0$, *there is* $g \in \mathcal{H}(P_1, P_2)$ *such that* $\hat{d}(h, g) < \varepsilon$ *and* $g(\ker\mathcal{A}) = \ker\mathcal{B}$.

PROOF: Observe that (a) is a triviality. For (b), let $h_1 : Q \to P_1$ be a homeomorphism. Then $\mathcal{C} = h_1^{-1}(\mathcal{A})$ and $\mathcal{D} = (h \circ h_1)^{-1}(\mathcal{B})$ are Q-matrices in Q. Since $h \circ h_1$ is uniformly continuous, there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d((h \circ h_1)(x), (h \circ h_1)(y)) < \varepsilon/2$. By theorem 3.1.12 (b) there is $\alpha \in \mathcal{H}(Q)$ such that $\hat{d}(\alpha, 1) < \delta$ and $\alpha(\ker\mathcal{C}) = \ker\mathcal{D}$. Let $g = h \circ h_1 \circ \alpha \circ h_1^{-1}$. Then $g : P_1 \to P_2$ is a homeomorphism and $g(\ker\mathcal{A}) = \ker\mathcal{B}$. Furth-

ermore by lemma 3.1.2 and the choice of δ,

$$\begin{aligned}\hat{d}(h, g) &= \hat{d}(h, h \circ h_1 \circ \alpha \circ h_1^{-1}) \\ &= \hat{d}(h \circ h_1, h \circ h_1 \circ \alpha) < \varepsilon. \ \square\end{aligned}$$

Van Mill used theorem 3.1.12 to prove that if X is a non-locally compact countable space, then $C_p^*(X)$ is homeomorphic to σ_ω. The strategy of the proof is the following: First a test space T of X is constructed and a Q-matrix $\mathcal{B}$ is found such that $\ker \mathcal{B} = C_p^*(T)$. So by theorem 3.1.12 (a) it follows that $C_p^*(T)$ is homeomorphic to σ_ω. Then by applying strong results of Toruńczyk [50], [51] he derives that $C_p^*(X)$ is homeomorphic to σ_ω. In section 3.3 we will use the same strategy to prove that $C_p(X)$ is homeomorphic to σ_ω.

3.1.14 EXAMPLE: Let Σ_n and Σ be as in example 3.1.8. Let $P = \prod_{i=1}^{\infty} Q_i$, where $Q_i = Q$ for every $i \in \mathbb{N}$. Clearly P is a Hilbert cube. For every $n, m \in \mathbb{N}$ define $A_m^n \subset P$ as follows

(1) $A_1^n = \emptyset$ for every $n \in \mathbb{N}$ and
(2) $A_m^n = (\Sigma_m)^n \times Q \times Q \times Q \times \cdots$ for every $n \in \mathbb{N}$ and $m \geq 2$.

We claim that $\mathcal{A} = \{A_m^n : n, m \in \mathbb{N}\}$ is a Q-matrix in P. By lemma 3.1.4 for each $n, m \in \mathbb{N}$, $A_m^n \in \mathcal{I}(P)$ and $A_m^n \in \mathcal{I}(A_{m+1}^n)$. For each $n \in \mathbb{N}$, A_m^n is convex and infinite-dimensional, and $\bigcup_{m=1}^{\infty} A_m^n$ is dense in P, so by theorem 3.1.7 we have that $\{A_m^n\}_{m>1}$ is a skeleton in P for every $n \in \mathbb{N}$. Now let $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$. By the observation made above we may assume $i_1 < \cdots < i_m$. Then

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} = (\Sigma_{i_1})^{n_1} \times (\Sigma_{i_2})^{n_2 - n_1} \times \cdots \times (\Sigma_{i_m})^{n_m - n_{m-1}} \times Q \times Q \times Q \times \cdots$$

is a product of Hilbert cubes and hence a Hilbert cube itself.

For $p \in \mathbb{N}$ and $i \geq i_m$,

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} = (\Sigma_{i_1})^{n_1} \times \cdots \times (\Sigma_{i_m})^{n_m - n_{m-1}} \times (\Sigma_i)^p \times Q \times Q \times Q \times \cdots$$

and for $p \in \mathbb{N}$ and $i < i_m$, $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} = A_i^{n_m+p}$. By the above formulas and by lemma 3.1.4, for each $i \in \mathbb{N}$,

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k}) \text{ and } \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}).$$

Furthermore we have that $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is convex and infinite-dimensional, and

$\bigcup_{i=1}^{\infty}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p})$ is dense in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$. So again by theorem 3.1.7, for every $p \in \mathbb{N}$, $\{\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$.

Finally let $s, t \in \mathbb{N}$ be such that $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subset A_t^s$. Then by the above formulas we can write for $t > 1$

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} = \prod_{i=1}^{\infty} E_i \text{ and } \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s = \prod_{i=1}^{\infty} F_i,$$

where all but finitely many E_i's and F_i's are equal to Q and the remaining finitely many E_i's and F_i's are elements of the family $\{\Sigma_n\}_{n>1}$. Since $\Pi_{i=1}^{\infty} F_i$ is a proper subset of $\Pi_{i=1}^{\infty} E_i$, there is $i \in \mathbb{N}$ such that F_i is a proper subset of E_i. Since $F_i = \Sigma_n$ for some $n \in \mathbb{N}$ or Q and F_i also, it follows that each factor space of F_i is a proper subset of the corresponding factor space of E_i. Hence by lemma 3.1.4 it follows that $\Pi_{i=1}^{\infty} F_i \in \mathcal{I}(\Pi_{i=1}^{\infty} E_i)$.

It is easily seen that $\ker \mathcal{A} = \Sigma^{\infty}$, so that by theorem 3.1.12 (a) Σ^{∞} is homeomorphic to σ_{ω}.

COROLLARY 3.1.15: *Let* $\{P_i : i \in \mathbb{N}\}$ *be a family of Hilbert cubes, and* $P = \Pi_{i=1}^{\infty} P_i$. *Then*

(a) *if* $\mathcal{B}$ *and* $\mathcal{C}$ *are Q-matrices in* P_1 *resp.* P_2, *then there is a Q-matrix* $\mathcal{A}$ *in* $P_1 \times P_2$ *such that* $\ker \mathcal{A} = \ker \mathcal{B} \times \ker \mathcal{C}$,

(b) *if for each* $i \in \mathbb{N}$, $\mathcal{A}_i$ *is a Q-matrix in* P_i, *then there is a Q-matrix* $\mathcal{A}$ *in* P *such that* $\ker \mathcal{A} = \Pi_{i=1}^{\infty} \ker \mathcal{A}_i$,

(c) *if for each* $i \in \mathbb{N}$, A_i *is a skeletoid in* P_i, *then there is a Q-matrix* $\mathcal{A}$ *in* P *such that* $\ker \mathcal{A} = \Pi_{i=1}^{\infty} A_i$,

(d) *if* $\mathcal{B}$ *is a Q-matrix in* P_1 *and* A *is a skeletoid in* P_2, *then there is a Q-matrix* $\mathcal{A}$ *in* $P_1 \times P_2$ *such that* $\ker \mathcal{A} = \ker \mathcal{B} \times A$, *and*

(e) *if* $\mathcal{B}$ *is a Q-matrix in* P_1, *then there is a Q-matrix* $\mathcal{A}$ *in* $[-1, 1] \times P_1$ *such that* $\ker \mathcal{A} = (-1, 1) \times \ker \mathcal{B}$.

PROOF: If $\mathcal{A}$ is a Q-matrix in a Hilbert cube Q_1, there is by corollary 3.1.13 (b) and example 3.1.14, a homeomorphism $h : Q^{\infty} \to Q_1$ such that $h(\Sigma^{\infty}) = \ker \mathcal{A}$. Moreover by theorem 3.1.9 and example 3.1.8, there is for each skeletoid A in a Hilbert cube Q_1 a homeomorphism $g : Q \to Q_1$ such that $g(\Sigma) = A$. It is easily seen that

(1) there is a homeomorphism $h_1 : Q^{\infty} \to (Q^{\infty})^2$ such that $h_1(\Sigma^{\infty}) = (\Sigma^{\infty})^2$,

(2) there is a homeomorphism $h_2 : Q^{\infty} \to (Q^{\infty})^{\infty}$ such that $h_2(\Sigma^{\infty}) = (\Sigma^{\infty})^{\infty}$,

(3) there is a homeomorphism $h_3 : Q^{\infty} \to Q \times Q^{\infty}$ such that $h_3(\Sigma^{\infty}) = \Sigma \times \Sigma^{\infty}$,

and

(4) there is a homeomorphism $h_4: Q^\infty \to [-1, 1]\times Q^\infty$ such that $h_4(\Sigma^\infty) = (-1, 1)\times\Sigma^\infty$.

Taking the right combinations of the obtained homeomorphisms one can prove (a) through (e). We outline (e). The other proofs are similar.

Let $h: Q^\infty \to P_1$ be a homeomorphism such that $h(\Sigma^\infty) = \ker\mathcal{B}$. Define $\alpha: P_1 \to [-1, 1]\times P_1$ by $\alpha = (1\times h)\circ h_4 \circ h^{-1}$. Then α is clearly a well-defined homeomorphism. Furthermore

$$\begin{aligned}\alpha(\ker\mathcal{B}) &= ((1\times h)\circ h_4\circ h^{-1})(\ker\mathcal{B})\\ &= ((1\times h)\circ h_4)(\Sigma^\infty)\\ &= (1\times h)((-1, 1)\times\Sigma^\infty)\\ &= (-1, 1)\times\ker\mathcal{B}.\end{aligned}$$

Then $\mathcal{A} = \alpha(\mathcal{B})$ is a Q-matrix in $[-1, 1]\times P_1$ such that $\ker(\mathcal{A}) = (-1, 1)\times\ker\mathcal{B}$. □

In contrast to the theory of skeletoids, the theory of Q-matrices is hardly developed. In view of theorem 3.1.10, a first question to ask, is whether for a Q-matrix $\mathcal{A}$ in a Hilbert cube P and $F \in \mathcal{I}(P)$, there is a Q-matrix $\mathcal{B}$ such that $\ker\mathcal{B} = \ker\mathcal{A}\setminus F$. We were not able to prove this straight from the definition of a Q-matrix. However we can prove the weaker statement that $\ker\mathcal{A}$ and $\ker\mathcal{A}\setminus F$ are homeomorphic (theorem 3.1.21). As will be clear in the sequel, the proof of this statement unfortunately has nothing to do with Q-matrices. Before the proof can be given we have to present some more definitions and known theorems.

A space X is said to be a ***σ_ω-manifold*** if there is an open cover of X consisting of sets homeomorphic to open subsets of σ_ω. Two spaces X and Y have the same ***homotopy type*** whenever there are $f \in C(X, Y)$ and $g \in C(Y, X)$ such that $f\circ g$ is homotopic to 1_Y and $g\circ f$ is homotopic to 1_X. We have the following theorem of Henderson.

3.1.16 THEOREM ([30]): *If X and Y are σ_ω-manifolds, then X is homeomorphic to Y if and only if X and Y have the same homotopy type.*

A space X is an ***absolute retract*** (abbreviated AR), resp. an ***absolute neigborhood retract*** (abbreviated ANR), whenever for every space Y and for every closed subspace A of Y, every continuous function $f: A \to X$ has an extension $\tilde{f}: Y \to X$, resp. an extension $\tilde{f}: U \to X$ over a neigborhood U of A in Y. By the Dugundji Extension Theorem, σ_ω is an AR. A space X which admits an open cover consisting of ANR's, is itself an

ANR ([39, Th. 5.4.5]). A space X is ***contractible*** whenever the identity 1_X is homotopic to a constant mapping. Note that two contractible spaces have the same homotopy type. A space X is ***homotopically trivial*** whenever for every $n \in \mathbb{N}$ and every continuous $f: S^n \to X$, there is a continuous extension $g: B^{n+1} \to X$. The above notions are related by the following

3.1.17 THEOREM ([39, Th. 5.2.15]): *For a space X are equivalent*

(1) *X is an AR,*

(2) *X is an ANR and contractible, and*

(3) *X is an ANR and homotopically trivial.*

We proceed by proving the announced statement from the previous page (cf. Theorem 3.1.21). We start with three lemmas.

3.1.18 LEMMA: *Let $K \subset \Sigma$ be compact. Then for every $\varepsilon > 0$, there is an embedding $f: Q \to \Sigma$ such that*

(a) $f \mid K = 1$, and

(b) $\hat{d}(f, 1) < \varepsilon$

PROOF: By lemma 3.1.4, $K \in \mathcal{I}(Q)$ and by example 3.1.8, $\{\Sigma_n\}_{n \in \mathbb{N}}$ is a skeleton in Q, so there are $n \in \mathbb{N}$ and $h \in \mathcal{H}(Q)$ such that

(1) $h(K) \subset \Sigma_n$, and

(2) $\hat{d}(h, 1) < \varepsilon/6$.

By lemma 3.1.4, $h(K) \in \mathcal{I}(Q)$. By theorem 3.1.10, $\Sigma \setminus h(K)$ is a skeletoid in Q. Furthermore $h(\Sigma \setminus K)$ is a skeletoid in Q and $h(K)$ misses $\Sigma \setminus h(K)$ and $h(\Sigma \setminus K)$ so that by theorem 3.1.9, there is $\alpha \in \mathcal{H}(Q)$ such that

(3) $\alpha(h(\Sigma \setminus K)) = \Sigma \setminus h(K)$,

(4) $\alpha \mid h(K) = 1$, and

(5) $\hat{d}(\alpha, 1) < \varepsilon/6$.

Since $h(K) \subset \Sigma$, we have by (3) and (4) that α has the additional property that $\alpha(h(\Sigma)) = \Sigma$. Let $\beta = \alpha \circ h$. Find $m \in \mathbb{N}$ and a homeomorphism $\xi: Q \to \Sigma_m$ such that

(6) $\xi \mid \Sigma_n = 1$, and

(7) $\hat{d}(\xi, 1) < \varepsilon/3$.

Let $f = \beta^{-1} \circ \xi \circ \beta: Q \to \Sigma$. Then f is clearly a well-defined embedding. Furthermore for

$x \in K$,

$$\begin{aligned} f(x) &= (\beta^{-1} \circ \xi \circ \alpha \circ h)(x) \\ &= (\beta^{-1} \circ \xi \circ h)(x) \quad \text{by (4)} \\ &= (h^{-1} \circ \alpha^{-1} \circ h)(x) \quad \text{by (1) and (6)} \\ &= x \quad \text{by (4),} \end{aligned}$$

and by (2), (5), (7), and lemma 3.1.2,

$$\begin{aligned} \hat{d}(f, 1) &\le \hat{d}(\beta^{-1} \circ \xi \circ \beta, \xi \circ \beta) + \hat{d}(\xi \circ \beta, \beta) + \hat{d}(\beta, 1) \\ &\le \hat{d}(\beta^{-1}, 1) + \hat{d}(\xi, 1) + \hat{d}(\beta, 1) \\ &= 2\hat{d}(\beta, 1) + \hat{d}(\xi, 1) \\ &\le 2\hat{d}(\alpha, 1) + 2\hat{d}(h, 1) + \hat{d}(\xi, 1) < \varepsilon. \ \square \end{aligned}$$

3.1.19 LEMMA: *Let $K \subset \Sigma^\infty$ be compact. Then for every $\varepsilon > 0$, there is an embedding $f : Q^\infty \to \Sigma^\infty$ such that*

(a) $f \mid K = 1$, and

(b) $\hat{d}(f, 1) < \varepsilon$.

PROOF: For every $n \in \mathbb{N}$, $\pi_n(K) \subset \Sigma$, so by lemma 3.1.18, there is an embedding $f_n : Q \to \Sigma$ such that $f_n \mid \pi_n(K) = 1$ and $\hat{d}(f_n, 1) < \varepsilon$. Define $f : Q^\infty \to \Sigma^\infty$ by $f = (f_1, f_2, \ldots)$ Then f is easily seen to be as required. $\square$

3.1.20 LEMMA: *Let $K \subset \Sigma^\infty$ be compact and $Z \in \mathcal{Z}(Q^\infty)$ such that $K \cap Z = \emptyset$. Then for every $\varepsilon > 0$ there is an embedding $f : Q^\infty \to \Sigma^\infty$ such that*

(a) $f \mid K = 1$,

(b) $f(Q^\infty) \cap Z = \emptyset$, and

(c) $\hat{d}(f, 1) < \varepsilon$.

PROOF: There is a continuous $h_1 : Q^\infty \to Q^\infty$ such that $h_1(Q^\infty) \cap Z = \emptyset$ and $\hat{d}(h_1, 1) < \varepsilon/8$. Let $\eta = d(h_1(Q^\infty), Z)$. By theorem 3.1.6 there is a Z-embedding $h_2 : Q^\infty \to Q^\infty$ such that $\hat{d}(h_2, h_1) < \min\{\eta, \varepsilon/8\}$. Then $h_2(Q^\infty) \in \mathcal{Z}(Q^\infty)$ and $h_2(Q^\infty) \cap Z = \emptyset$.

Define $g : h_2(K) \cup Z \to K \cup Z$ by

$$g(x) = \begin{cases} h_2^{-1}(x) & \text{if } x \in h_2(K) \\ x & \text{if } x \in Z \end{cases}.$$

Then g is a homeomorphism between Z-sets in Q^∞ such that $\hat{d}(g, 1_{h_2(K)\cup Z}) < \varepsilon/4$, so by theorem 3.1.5, g extends to a homeomorphism $\bar{g}: Q^\infty \to Q^\infty$ such that $\hat{d}(\bar{g}, 1) < \varepsilon/4$. Let $h = \bar{g} \circ h_2$. Then h satisfies

(1) $h \mid K = 1$,
(2) $h(Q^\infty) \cap Z = \emptyset$, and
(3) $\hat{d}(h, 1) < \varepsilon/2$.

Let $\xi = d(h(Q^\infty), Z)$. By lemma 3.1.19, there is an embedding $\alpha: Q^\infty \to \Sigma^\infty$ such that $\alpha \mid K = 1$ and $\hat{d}(\alpha, 1) < \min\{\varepsilon/2, \xi\}$. Let $f = \alpha \circ h$. Then f is easily seen to be as required. □

We are now in a position to prove the announced

3.1.21 THEOREM: *Let $\mathcal{A} = \{A_m^n : n, m \in \mathbb{N}\}$ be a Q-matrix in a Hilbert cube P and $Z \in \mathcal{Z}(P)$. Then* $\ker \mathcal{A}$ *and* $\ker \mathcal{A} \setminus Z$ *are homeomorphic.*

PROOF: By corollary 3.1.13 (b) and example 3.1.14, there is a homeomorphism $h: P \to Q^\infty$ such that $h(\ker \mathcal{A}) = \Sigma^\infty$. It suffices to prove that for $Z \in \mathcal{Z}(Q^\infty)$, $\Sigma^\infty \setminus Z$ is homeomorphic to Σ^∞.

By corollary 3.1.13 (a), Σ^∞ is homeomorphic to σ_ω, so Σ^∞ is an AR. Obviously $\Sigma^\infty \setminus Z$ is a σ_ω-manifold and hence an ANR.

CLAIM: $\Sigma^\infty \setminus Z$ is homotopically trivial.

Let $f: S^n \to \Sigma^\infty \setminus Z$ be a continuous function. Then since Σ^∞ is an AR, f extends to a continuous function $g_1: B^{n+1} \to \Sigma^\infty$. Then $f(S^n)$ is a compact subset of Σ^∞ such that $f(S^n) \cap Z = \emptyset$. By lemma 3.1.20 we then have an embedding $g_2: Q^\infty \to \Sigma^\infty$ such that $g_2 \mid f(S^n) = 1$ and $g_2(Q^\infty) \cap Z = \emptyset$. Let $g = g_2 \circ g_1: B^{n+1} \to \Sigma^\infty \setminus Z$. Then g is easily seen to be an extension of f.

By theorem 3.1.17 and the claim we now have that Σ^∞ and $\Sigma^\infty \setminus Z$ are both contractible and hence they have the same homotopy type. By theorem 3.1.16 we then have that Σ^∞ is homeomorphic to $\Sigma^\infty \setminus Z$. □

We finish this section with the remark that we did not prove all results in this section. Their proofs are beyond the scope of this monograph. For more information on infinite-dimensional topology we refer to [11], [17], [18] and [39]. For more information on AR theory we refer to [14], [31] and [39].

§3.2. Homeomorphic function spaces part 1

Let $X=\{x_0, x_1, x_2, \ldots\}$ be a countable space which fails to be locally compact at x_0. We shall prove that $C_{p,0}(X)$ can be written as the kernel of a Q-matrix in some Hilbert cube.

Let $\psi\colon \mathbb{N}\to X\times\mathbb{N}$ be a bijection and define $\phi\colon \mathbb{N}\to X$ by $\phi=\pi_1\circ\psi$, where $\pi_1\colon X\times\mathbb{N}\to X$ is the projection. The following lemma is of fundamental importance in the process of describing $C_{p,0}(X)$ as the kernel of a Q-matrix.

3.2.1 LEMMA: *There exists a decreasing clopen base* $\{U_i^{x_0}\}_{i\in\mathbb{N}}$ *at* x_0 *and for each* $x\neq x_0$ *there exists a clopen neighborhood* U^x *of* x *such that for every* $n\in\mathbb{N}$,

(*a*) *if* $\phi(n)\neq x_0$, *then* $U^{\phi(n)}\cap U_n^{x_0}=\emptyset$, *and*

(*b*) *for* $s, n\in\mathbb{N}$, *we have* $U_s^{x_0}\setminus[\,U_{s+1}^{x_0}\cup\bigcup\{U^{\phi(j)}: j\leq n,\ \phi(j)\neq x_0\}\,]$ *is infinite.*

PROOF: Since no neighborhood of x_0 is compact, there exists a decreasing clopen base $\{V_i^{x_0}\}_{i\in\mathbb{N}}$ at x_0 such that for every $i\in\mathbb{N}$, $V_i^{x_0}\setminus V_{i+1}^{x_0}$ contains an infinite closed discrete subset D_i.

We construct inductively a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ of natural numbers and for each $n\in\mathbb{N}$ such that $\phi(n)\neq x_0$ a clopen neighborhood V_n of $\phi(n)$ satisfying

(1) $V_n\cap V_{i_n}^{x_0}=\emptyset$, and

(2) $V_n\cap\bigcup_{s<i_n}D_s$ contains at most one point.

Suppose we found for $n\in\mathbb{N}$, $i_1,\ldots,i_{n-1}$ and V_j for $j<n$ such that $\phi(j)\neq x_0$. If $\phi(n)=x_0$ let $i_n>i_{n-1}$ be arbitrary. If $\phi(n)=\phi(j)\neq x_0$ for some $j<n$, let $V_n=V_j$ and $i_n>i_{n-1}$ arbitrary. Since we deal with a decreasing base at x_0 we have $V_n\cap V_{i_n}^{x_0}=\emptyset$ and $V_n\cap\bigcup_{s\geq i_j}D_s=\emptyset$, hence also $V_n\cap\bigcup_{s<i_n}D_s$ contains at most one point. If $\phi(n)\notin\{\phi(j): j<n,\ \phi(j)\neq x_0\}\cup\{x_0\}$, we can find a clopen neighborhood U of $\phi(n)$ and $i_n>i_{n-1}$ such that $U\cap V_{i_n}^{x_0}=\emptyset$. Since $\bigcup_{s<i_n}D_s$ is closed and discrete we can find a clopen neighborhood V_n of $\phi(n)$ contained in U such that $V_n\cap\bigcup_{s<i_n}D_s$ contains at most one point. This completes the inductive construction.

For $n\in\mathbb{N}$, let $U_n^{x_0}=V_{i_n}^{x_0}$. For $x\neq x_0$, let $k(x)=\min\phi^{-1}(x)$, and let $U^x=V_{k(x)}$. Let $n\in\mathbb{N}$ with $\phi(n)\neq x_0$. Then $k(\phi(n))\leq n$, so by (1), $U^{\phi(n)}\cap U_n^{x_0}=\emptyset$. In addition we have that for $s\geq i_n$, $U^{\phi(n)}\cap D_s=\emptyset$. Hence by (2) we have for $s, n\in\mathbb{N}$ that

$U_s^{x_0} \setminus [\, U_{s+1}^{x_0} \cup \bigcup \{U^{\phi(j)} : j \leq n,\ \phi(j) \neq x_0\}\,]$ is infinite. This proves the lemma. □

As mentioned in the introduction of this chapter, $C_p(X) = \{f : X \to (-1, 1) : f \text{ is continuous}\}$. Recall that $C_{p,0}(X) = \{f \in C_p(X) : f(x_0) = 0\}$. We look at these spaces as subspaces of $[-1, 1]^X$ endowed with the product topology. $[-1, 1]^X$ is obviously a Hilbert cube. Recall that on $[-1, 1]^X$ we use the metric

$$d(f, g) = \sum_{i=0}^{\infty} 2^{-i} |f(x_i) - g(x_i)|,$$

for $f, g \in [-1, 1]^X$.

We will now give another description of the space $C_{p,0}(X)$, in terms of the kernel of a Q-matrix. Let $Y = X \setminus \{x_0\}$ and $P = \{0\} \times [-1, 1]^Y$. Evidently there is a convexity preserving homeomorphism between P and Q. Hence by theorem 3.1.1 each Keller space in P is a Hilbert cube. Let $\{U_n^{x_0}\}_{n \in \mathbb{N}}$ and $\{U^x : x \neq x_0\}$ be as in lemma 3.2.1. For $x \neq x_0$ let $\{U_n^x\}_{n \in \mathbb{N}}$ be a clopen decreasing base at x such that $U_1^x = U^x$. For every $x \in X$ and $n, m \in \mathbb{N}$, we define

$$\text{(A) } B_m^{(x,n)} = \{\, g \in P : g(U_m^x) \subset [\, g(x) - \frac{1}{n},\ g(x) + \frac{1}{n}\,]\}.$$

Furthermore for every $n, m \in \mathbb{N}$ we define

$$\text{(B) } C_m^n = B_m^{(x_0, n)} \cap \bigcap_{j=1}^{n} B_m^{(\phi(j), n)},$$

$$\text{(C) } L_m^n = \{0\} \times [-1 + \frac{1}{m},\ 1 - \frac{1}{m}]^{\{x_1, \ldots, x_n\}} \times [-1, 1] \times [-1, 1] \times \cdots, \text{ and}$$

$$\text{(D) } A_1^n = \emptyset, \text{ and } A_m^n = C_m^n \cap L_m^n \text{ for } m \geq 2.$$

It will turn out that the family $\mathcal{A} = \{A_m^n : n, m \in \mathbb{N}, n > 1\}$ is a Q-matrix in P such that $\ker \mathcal{A} = C_{p,0}(X)$.

3.2.2 LEMMA: *For every $n, m \in \mathbb{N}$, we have*

(a) A_m^n is closed in P,

(b) $A_m^n \subset A_{m+1}^n$, and

(c) $A_m^{n+1} \subset A_m^n$.

PROOF: It is easily seen that for every $x \in X$ and for every $n, m \in \mathbb{N}$ we have,

(1) $B_m^{(x,n)}$ and L_m^n are closed in $[-1, 1]^X$,

(2) $B_m^{(x,n)} \subset B_{m+1}^{(x,n)}$, $B_m^{(x,n+1)} \subset B_m^{(x,n)}$,

(3) $L_m^n \subset L_{m+1}^n$ and $L_m^{n+1} \subset L_m^n$.

Now the lemma follows from (1), (2) and (3). □

We first prove that $\ker \mathcal{A} = C_{p,0}(X)$ (lemma 3.2.4). Define $F = \{f \in [-1, 1]^X : f$ is continuous and $f(x_0) = 0\}$. Then $F \subset P$. Observe that by the definition of continuity, $F = \bigcap_{x \in X} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B_m^{(x,n)}$ (see [19]).

3.2.3 LEMMA: $F = \bigcap_{n=2}^{\infty} \bigcup_{m=1}^{\infty} C_m^n$.

PROOF: First suppose that $f \in F$ and $n \geq 2$. Then $f \in P$. Because f is continuous, there exists $m \in \mathbb{N}$ such that for each $j \leq n$,

$$f(U_m^{\phi(j)}) \subset [f(\phi(j)) - \frac{1}{n}, f(\phi(j)) + \frac{1}{n}],$$

and

$$f(U_m^{x_0}) \subset [-\frac{1}{n}, \frac{1}{n}].$$

This implies $f \in B_m^{(x_0,n)} \cap \bigcap_{j=1}^{n} B_m^{(\phi(j),n)} = C_m^n$. We conclude that $f \in \bigcap_{n=2}^{\infty} \bigcup_{m=1}^{\infty} C_m^n$.

Secondly suppose $f \in \bigcap_{n=2}^{\infty} \bigcup_{m=1}^{\infty} C_m^n$. Let $x \in X$ and $n \in \mathbb{N}$. Because ψ is a bijection, there exists $n_x \in \mathbb{N}$ such that $k = \psi^{-1}(x, n_x) > n$. Then $\phi(k) = x$. There exists $m \in \mathbb{N}$ such that $f \in C_m^k$, hence

$$f \in C_m^k \subset B_m^{(x,k)} \subset B_m^{(x,n)}.$$

So $f \in \bigcap_{x \in X} \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B_m^{(x,n)} = F$. This completes the proof of the lemma. □

3.2.4 LEMMA: $C_{p,0}(X) = \bigcap_{n=2}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$.

PROOF: First suppose $f \in C_{p,0}(X)$ and $n \geq 2$. There exists $k \geq 2$ such that

$$\{f(x_1), \ldots, f(x_n)\} \subset [-1 + \frac{1}{k}, 1 - \frac{1}{k}],$$

hence $f \in L_k^n$. Since $f \in F$, there exists $m \geq k$ such that $f \in C_m^n$. So $f \in C_m^n \cap L_m^n = A_m^n$. We conclude that $f \in \bigcap_{n=2}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$.

Secondly suppose $f \in \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$. Since for every $n, m \in \mathbb{N}$, $A_m^n \subset C_m^n$, we have by lemma 3.2.3, $f \in F$. Since for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $f \in A_m^n \subset L_m^n$, we have that $\{f(x_1), \ldots, f(x_n)\} \subset (-1, 1)$. So $f \in C_{p,0}(X)$. This proves the lemma. □

3.2.5 LEMMA: *For every $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$ we have that $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Keller space in P, hence a Hilbert cube.*

PROOF: To prove that $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Keller space, we have to verify that $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is compact, convex and infinite-dimensional.

By lemma 3.2.2 (a), $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is closed in P, hence it is compact. To prove that $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is convex we first claim that for every $x \in X$ and for every $n, m \in \mathbb{N}$, $B_m^{(x,n)}$ is convex. Indeed, let $f, g \in B_m^{(x,n)}$, $\lambda \in [0, 1]$ and $h = \lambda f + (1-\lambda)g$. Since P is convex, we have $h \in P$. Furthermore if $y \in U_m^x$, then

$$|h(x) - h(y)| \leq \lambda |f(x) - f(y)| + (1-\lambda)|g(x) - g(y)|$$
$$\leq \lambda \frac{1}{n} + (1-\lambda)\frac{1}{n} = \frac{1}{n}.$$

So $h \in B_m^{(x,n)}$, so $B_m^{(x,n)}$ is convex. It is easily seen that for every $n, m \in \mathbb{N}$, L_m^n is convex. Since the intersection of convex sets is again convex, $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is convex.

Finally, to see that $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is infinite-dimensional notice that for every $x \in X$ and for every $n, m \in \mathbb{N}$,

$$\{0\} \times [-\frac{1}{2n}, \frac{1}{2n}]^Y \subset B_m^{(x,n)}.$$

and for every $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{1\}$, $\{0\} \times [-\frac{1}{2n}, \frac{1}{2n}]^Y \subset L_m^n$, so

$$\{0\} \times [-\frac{1}{2n_m}, \frac{1}{2n_m}]^Y \subset \bigcap_{k=1}^{m} A_{i_k}^{n_k}.$$

We conclude that $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Keller space in P, hence by theorem 3.1.1 $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Hilbert cube. □

For every $i, k \in \mathbb{N}$ we define $h(i, k): P \to P$ by

$$h(i, k)(f)(x) = \begin{cases} f(x) & \text{if } x \neq x_i \\ \frac{1}{k} & \text{if } x = x_i \end{cases}$$

Notice that since $x_i \neq x_0$, $h(i, k)$ is well-defined.

3.2.6 LEMMA: *Let $i, k \in \mathbb{N}$. Then*

(a) $h(i, k)$ is continuous and $\hat{d}(h(i, k), 1) \leq 2^{-i+1}$, and

(b) if $i > 1$, $k \leq p < i$ and $s \in \mathbb{N}$ are such that

$x_i \in U_s^{x_0} \setminus [\, U_{s+1}^{x_0} \cup \bigcup \{ U_1^{\phi(j)} : j \leq p,\ \phi(j) \neq x_0 \} \,]$, *then*

(i) for $n \leq k$ and $m \in \mathbb{N}$, $h(i, k)(A_m^n) \subset A_m^n$,

(ii) for $n \leq p$ and $m > s$, $h(i, k)(A_m^n) \subset A_m^n$, and

(iii) for $n > k$, $h(i, k)(P) \cap A_s^n = \emptyset$.

PROOF: For every $x \in X$, let $\pi_x : P \to [-1, 1]$ be the projection onto the x-th coordinate. Then for $x \neq x_i$, we have $\pi_x \circ h(i, k) = \pi_x$ and for $x = x_i$, we have $\pi_x \circ h(i, k) \equiv 1/k$, so that for each $x \in X$, $\pi_x \circ h(i, k)$ is continuous. So we have that $h(i, k)$ is continuous. Furthermore we have

$$\begin{aligned}
\hat{d}(h(i, k), 1) &= \sup \{ d(h(i, k)(f), f) : f \in P \} \\
&= \sup \{ \Sigma_{j=1}^{\infty} 2^{-j} \, | h(i, k)(f)(x_j) - (f)(x_j) | : f \in P \} \\
&= \sup \{ 2^{-i} \, | 1/k - f(x_i) | : f \in P \} \\
&\leq 2^{-i} \cdot 2 = 2^{-i+1}
\end{aligned}$$

We prove (b)(i) and (b)(ii) simultaneously. Let $n, m \in \mathbb{N}$ be such that $n \leq k$ and $m \in \mathbb{N}$ or such that $n \leq p$ and $m > s$. If $m = 1$ there is nothing to prove, so let $m > 1$ and $f \in A_m^n$. We have to prove that $h(i, k)(f) \in A_m^n = L_m^n \cap C_m^n$.

Since $f \in L_m^n$ and $n \leq p < i$, we have for $j \leq n$

$$h(i, k)(f)(x_j) = f(x_j) \in [-1 + \frac{1}{m}, 1 - \frac{1}{m}].$$

Hence $h(i, k)(f) \in L_m^n$. To prove that $h(i, k)(f) \in C_m^n$, we take $y \in U_m^x$, where $x \in \{x_0\} \cup \{\phi(j) : j \leq n\}$. Notice that $x \neq x_i$ because

$$x_i \notin U_{s+1}^{x_0} \cup \bigcup \{ U_1^{\phi(j)} : j \leq p,\ \phi(j) \neq x_0 \},$$

and $n \leq p$.

If $y \neq x_i$, then since $f \in B_m^{(x, n)}$,

$$| h(i, k)(f)(y) - h(i, k)(f)(x) | = | f(y) - f(x) | \leq \frac{1}{n}.$$

Now assume that $y = x_i$. If $x \neq x_0$, then $x = \phi(j)$ for some $j \leq n$ with $\phi(j) \neq x_0$. Then

$$x_i = y \in U_m^{\phi(j)} \subset U_1^{\phi(j)},$$

which is a contradiction. Hence $x = x_0$. If $m > s$, then $U_m^{x_0} \subset U_{s+1}^{x_0}$, so $x_i \notin U_m^{x_0}$. But this means that $y \notin U_m^x$, which gives a contradiction. So $n \leq k$ and we get

$$|h(i, k)(f)(y) - h(i, k)(f)(x)| = |\frac{1}{k} - 0| = \frac{1}{k} \leq \frac{1}{n}.$$

We conclude that $h(i, k)(f) \in B_m^{(x, n)}$, hence

$$h(i, k)(f) \in B_m^{(x_0, n)} \cap \bigcap_{j=1}^n B_m^{(\phi(j), n)} = C_m^n.$$

So $h(i, k)(f) \in C_m^n \cap L_m^n = A_m^n$. This proves (b)(i) and (b)(ii).

For (b)(iii), let $n > k$. Since $x_i \in U_s^{x_0}$, we have for $f \in P$,

$$|h(i, k)(f)(x_i) - h(i, k)(f)(x_0)| = \frac{1}{k} > \frac{1}{n}.$$

We conclude that $h(i, k)(f) \notin B_s^{(x_0, n)}$, hence $h(i, k)(f) \notin A_s^n$. This proves the lemma. □

3.2.7 COROLLARY: *For every $n, m \in \mathbb{N}$ with $n > 1$ we have that $A_m^n \in \mathcal{I}(P)$ and $A_m^n \in \mathcal{I}(A_{m+1}^n)$.*

PROOF: By lemma 3.2.2, we have that A_m^n is closed in P and $A_m^n \subset A_{m+1}^n$, hence A_m^n is closed in A_{m+1}^n.

Let $f \in C(Q, P)$ or $f \in C(Q, A_{m+1}^n)$ and let $\varepsilon > 0$. By lemma 3.2.1 (b),

$$U_m^{x_0} \setminus [\, U_{m+1}^{x_0} \cup \bigcup \{ U_1^{\phi(j)} : j \leq n,\ \phi(j) \neq x_0 \} \,]$$

is infinite so we can choose x_i in this set such that $i > n$, and $2^{-i} < \varepsilon/2$. Define $g : Q \to P$ by $g = h(i, 1) \circ f$. Then g is well-defined. By lemma 3.2.6 (a), g is continuous and

$$\hat{d}(g, f) \leq \hat{d}(h(i, 1), 1) \leq 2^{-i+1} < \varepsilon.$$

If $f \in C(Q, A_{m+1}^n)$, we have by lemma 3.2.6 (b)(ii),

$$g(Q) \subset h(i, 1)(A_{m+1}^n) \subset A_{m+1}^n.$$

Furthermore because $n > 1$, we have by lemma 3.2.6 (b)(ii), $g(Q) \cap A_m^n = \emptyset$. We conclude that $A_m^n \in \mathcal{I}(P)$ and $A_m^n \in \mathcal{I}(A_{m+1}^n)$. □

3.2.8 COROLLARY: *For every $n_1 < \cdots < n_m \in \mathbb{N}$, $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$, $p \in \mathbb{N}$ and $i \in \mathbb{N}$ we have that*

$$\bigcap_{k=1}^m A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^m A_{i_k}^{n_k})$$

and

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}).$$

PROOF: By lemma 3.2.2, we have that $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is closed in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ and $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is closed in $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}$.

Let $f: Q \to \bigcap_{k=1}^{m} A_{i_k}^{n_k}$ or $f: Q \to \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}$ be continuous and let $\varepsilon > 0$. By lemma 3.2.1 (b),

$$U_i^{x_0} \setminus [\, U_{i+1}^{x_0} \cup \bigcup \{ U_1^{\phi(j)} : j \leq n_m + p,\ \phi(j) \neq x_0 \} \,]$$

is infinite, so we can choose x_{i_0} in this set, such that $i_0 > n_m+p$, and $2^{-i_0} < \varepsilon/2$. Define $g: Q \to P$ by $g = h(i_0, n_m) \circ f$. Then g is a well-defined continuous function and $\hat{d}(g, f) < \varepsilon$

Since $f(Q) \subset \bigcap_{k=1}^{m} A_{i_k}^{n_k}$, we have by lemma 3.2.6 (b)(i), $g(Q) \subset \bigcap_{k=1}^{m} A_{i_k}^{n_k}$. Furthermore if $f(Q) \subset A_{i+1}^{n_m+p}$, then by lemma 3.2.6 (b)(ii), $g(Q) \subset A_{i+1}^{n_m+p}$. By lemma 3.2.6 (b)(iii), $g(Q) \cap A_i^{n_m+p} = \emptyset$. We conclude that $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$ and $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p})$. □

3.2.9 LEMMA: *For every $n > 1$: $\{ A_m^n \}_{m>1}$ is a skeleton in P.*

PROOF: Observe that P is a Keller space. To prove the lemma we shall verify the conditions in theorem 3.1.7. By corollary 3.2.7 and lemma 3.2.2, $\{ A_m^n \}_{m>1}$ is an increasing family of Z-sets in P. Again by corollary 3.2.7, we have for every $m \in \mathbb{N}$, $A_m^n \in \mathcal{I}(A_{m+1}^n)$. By lemma 3.2.5, A_m^n is convex and infinite-dimensional. So we only have to verify that $\bigcup_{m=2}^{\infty} A_m^n$ is dense in P. Notice that $C_{p,0}(X)$ is dense in $\{0\} \times (-1, 1)^Y$ which is dense in P, hence $C_{p,0}(X)$ is dense in P. Since $\bigcup_{m=2}^{\infty} A_m^n \supset C_{p,0}(X)$, $\bigcup_{m=2}^{\infty} A_m^n$ is dense in P. We obtain that $\{A_m^n\}_{m>1}$ is a skeleton in P. □

3.2.10 LEMMA: *For every $n_1 < \cdots < n_m \in \mathbb{N}$, $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$ and $p \in \mathbb{N}$ we have that $\{ \bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$.*

PROOF: By lemma 3.2.5, $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Keller space. To prove the lemma we again shall verify the conditions in theorem 3.1.7. As mentioned in section 3.1 we may assume $i_1 < \cdots < i_m$. By corollary 3.2.8, we have for every $i \in \mathbb{N}$,

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k}),$$

and

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p}).$$

By lemma 3.2.5, $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is convex and infinite dimensional, so we only have to verify that $\bigcup_{i=1}^{\infty} A_i^{n_m+p} \cap \bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is dense in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$. To this end we have to prove that for arbitrary $g \in \bigcap_{k=1}^{m} A_{i_k}^{n_k}$, $y_1, \ldots, y_n \in X$ and $\varepsilon > 0$, we have for

$$U = \{ f \in P : |f(y_l) - g(y_l)| < \varepsilon \text{ for every } l \leq n \}$$

that $(U \cap \bigcap_{k=1}^{m} A_{i_k}^{n_k}) \cap \bigcup_{i=1}^{\infty} A_i^{n_m+p} \neq \emptyset$. Since we deal with decreasing clopen bases it is possible to find $i_0 > n_m + p$ such that $1/i_0 < \varepsilon$ and

(1) $U_{i_0}^{\phi(j_1)} \cap U_{i_0}^{\phi(j_2)} = \emptyset$ if $\phi(j_1) \neq \phi(j_2)$ and $j_1, j_2 \leq n_m+p$,

(2) $U_{i_0}^{\phi(j_0)} \cap U_{i_k}^{\phi(j)} = \emptyset$ if $\phi(j_0) \notin U_{i_k}^{\phi(j)}$, $k \leq m$, $j \leq n_k$ and $j_0 \leq n_m + p$,

(3) $U_{i_0}^{\phi(j_0)} \cap U_{i_k}^{x_0} = \emptyset$ if $\phi(j_0) \notin U_{i_k}^{x_0}$, $k \leq m$ and $j_0 \leq n_m + p$,

(4) for $a \in \{ \phi(j) : j \leq n_m+p \} \cup \{ x_0 \}$ and $y_l \neq a$ we have $y_l \notin U_{i_0}^{a}$ $(l \leq n)$, and

(5) for $\phi(j) \notin \{ x_0, .., x_{n_m} \}$ we have $U_{i_0}^{\phi(j)} \cap \{ x_0, .., x_{n_m} \} = \emptyset$ $(j \leq n_m+p)$.

Now define $f : X \to [-1, 1]$ by

$$f(x) = \begin{cases} (1-\frac{1}{i_0}) g(\phi(j)) & \text{if } x \in \bigcup_{j \leq n_m+p} U_{i_0}^{\phi(j)}, \\ 0 & \text{if } x \in U_{i_0}^{x_0}, \\ (1-\frac{1}{i_0}) g(x) & \text{elsewhere.} \end{cases}$$

By lemma 3.2.1 (a) and (1) there is for every $x \in X$ at most one $a \in \{ \phi(j) : j \leq n_m+p \} \cup \{ x_0 \}$ with $x \in U_{i_0}^{a}$, and since $g(X) \subset [-1, 1]$, $f(X) \subset [-1, 1]$, so we conclude that f is a well-defined mapping. Furthermore $f(x_0) = 0$, hence $f \in P$. To prove the claim we will show that $f \in (U \cap \bigcap_{k=1}^{m} A_{i_k}^{n_k}) \cap \bigcup_{i=1}^{\infty} A_i^{n_m+p}$. For that we first prove $f \in U$. To this end let $l \leq n$. If $y_l \notin \{ \phi(j) : j \leq n_m+p \} \cup \{ x_0 \}$ then by (4) and by definition of f, $f(y_l) = (1 - 1/i_0) g(y_l)$ so

$$|f(y_l)-g(y_l)| = \frac{1}{i_0}|g(y_l)| \leq \frac{1}{i_0} < \varepsilon.$$

If $y_l \in \{\phi(j): j \leq n_m+p\} \cup \{x_0\}$ then by the definition of f,

$$|f(y_l)-g(y_l)| \leq \frac{1}{i_0} < \varepsilon.$$

So we indeed have $f \in U$.

To prove that $f \in \bigcap_{k=1}^{m} A_{i_k}^{n_k}$, we first show that $f \in \bigcap_{k=1}^{m} C_{i_k}^{n_k}$. To this end let $k \leq m$ and $b \in \{\phi(j): j \leq n_k\} \cup \{x_0\}$. Notice that $b \in U_{i_0}^{b}$, hence $f(b) = (1-1/i_0)g(b)$. Let $x \in U_{i_k}^{b}$. First suppose there is $a \in \{\phi(l): l \leq n_m+p\} \cup \{x_0\}$ such that $x \in U_{i_0}^{a}$. In this case $f(x) = (1-1/i_0)g(a)$. If $a = x_0$ then by lemma 3.2.1 (a), $a = b \in U_{i_k}^{b}$, and if $a \neq x_0$ then by (2) and (3), $a \in U_{i_k}^{b}$. Since

$$g \in C_{i_k}^{n_k} \subset B_{i_k}^{(b,\, n_k)},$$

we now have

$$|f(x)-f(b)| = (1-\frac{1}{i_0})|g(a)-g(b)| \leq (1-\frac{1}{i_0})\frac{1}{n_k} < \frac{1}{n_k}.$$

Secondly suppose for every $a \in \{\phi(l): l \leq n_m+p\} \cup \{x_0\}$ we have $x \notin U_{i_0}^{a}$. Then

$$|f(x)-f(b)| = (1-\frac{1}{i_0})|g(x)-g(b)| \leq (1-\frac{1}{i_0})\frac{1}{n_k} < \frac{1}{n_k}$$

We conclude that $f \in \bigcap_{k=1}^{m} C_{i_k}^{n_k}$. Now let $k \leq m$ and $j \leq n_k$. By (5) we have

$$|f(x_j)| = |(1-\frac{1}{i_0})g(x_j)| \leq |g(x_j)| \leq 1-\frac{1}{i_k}.$$

So $f \in \bigcap_{k=1}^{m} C_{i_k}^{n_k} \cap \bigcap_{k=1}^{m} L_{i_k}^{n_k} = \bigcap_{k=1}^{m} A_{i_k}^{n_k}$.

Finally we have to prove that $f \in \bigcup_{i=1}^{\infty} A_i^{n_m+p}$. In fact we show that $f \in A_{i_0}^{n_m+p}$. Let $a \in \{\phi(j): j \leq n_m+p\} \cup \{x_0\}$, and let $x \in U_{i_0}^{a}$. Then $f(x) = (1-1/i_0)g(a) = f(a)$, so

$$|f(x)-f(a)| = 0 \leq \frac{1}{n_m+p}$$

So $f \in C_{i_0}^{n_m+p}$. Since for every $x \in X$, $|f(x)| \leq 1-1/i_0$ we have $f \in A_{i_0}^{n_m+p}$. This proves the lemma. □

3.2.11 LEMMA: *For every* $n_1 < \cdots < n_m \in \mathbb{N}$, $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$ *and* $s, t \in \mathbb{N}$ *such that* $s > 1$ *and* $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subset A_t^s$, *we have* $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$.

PROOF: Let $n_1 < \cdots < n_m \in \mathbb{N}$, $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$ and suppose $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \not\subset A_t^s$. As mentioned in section 3.1 we may assume $i_1 < \cdots < i_m$.

If $s > n_m$ we have by corollary 3.2.8,

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k}).$$

So from now on we assume that $s \leq n_m$. If there exists $l \leq m$ such that $s \leq n_l$ and $t \geq i_l$, then by lemma 3.2.2,

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \subset A_{i_l}^{n_l} \subset A_t^s$$

and we have a contradiction. So for every $k \leq m$, $s > n_k$ or $t < i_k$. There exists $r \leq m$ such that $n_{r-1} < s \leq n_r$. (Let $n_0 = 1$).

Let $f : Q \to \bigcap_{k=1}^{m} A_{i_k}^{n_k}$ and $\varepsilon > 0$. By lemma 3.2.1 (b),

$$U_t^{x_0} \setminus [\, U_{t+1}^{x_0} \cup \bigcup \{ U_1^{\phi(j)} : j \leq n_m, \phi(j) \neq x_0 \} \,]$$

is infinite. Choose x_i in this set, such that $i > n_m$, $2^{-i} < \varepsilon/2$. Notice that $x_i \notin \{ \phi(j) : j \leq n_m \}$. Define $g : Q \to P$ by $g = h(i, n_{r-1}) \circ f$. Then g is well-defined and $\hat{d}(f, g) < \varepsilon$. We claim that $g(Q) \subset \bigcap_{k=1}^{m} A_{i_k}^{n_k}$. To this end let $k \leq m$. If $n_k \leq n_{r-1}$, then by lemma 3.2.6 (b)(i),

$$h(i, n_{r-1})(A_{i_k}^{n_k}) \subset A_{i_k}^{n_k}.$$

If $n_{r-1} < n_k \leq n_m$, then $s \leq n_r \leq n_k$. So $t < i_k$. Then by lemma 3.2.6 (b)(ii),

$$h(i, n_{r-1})(A_{i_k}^{n_k}) \subset A_{i_k}^{n_k}.$$

So $g(Q) \in \bigcap_{k=1}^{m} A_{i_k}^{n_k}$.

To finish the proof of this lemma, notice that by lemma 3.2.6 (b)(iii), $g(Q) \cap A_t^s = \emptyset$ (because $s > n_{r-1}$), so that $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$. □

3.2.12 THEOREM: *$\mathcal{A}$ is a Q-matrix in P.*

PROOF: By lemma 3.2.2 and corollary 3.2.7, $\mathcal{A}$ is a $\mathcal{I}$-matrix. The theorem now follows directly from the lemmas 3.2.5, 3.2.9, 3.2.10 and 3.2.11. □

We now come to the main result of this section.

3.2.13 THEOREM: *Let X be a non-locally compact countable space. Then*

(a) $C_p(X)$ is homeomorphic to σ_ω, and

(b) If Y is another non-locally compact countable space, then for every $\varepsilon > 0$ there is $h \in \mathcal{H}(Q)$ such that $h(C_{p,0}(X)) = C_{p,0}(Y)$ and $\hat{d}(h, 1) < \varepsilon$.

PROOF: By lemma 3.2.2 and lemma 3.2.4 it follows that $C_{p,0}(X) = \ker \mathcal{A}$, so by corollary 3.1.13 (a), $C_{p,0}(X)$ is homeomorphic to σ_ω. By proposition 2.3.2, $C_p(X)$ is homeomorphic to $C_{p,0}(X) \times \mathbb{R}$, hence

$$C_p(X) \approx C_{p,0}(X) \times \mathbb{R} \approx \sigma_\omega \times \mathbb{R} \approx \sigma_\omega.$$

Now let Y be another non-locally compact countable space and let $\varepsilon > 0$. Then by corollary 3.1.13 (b), there is $h \in \mathcal{H}(Q)$ such that $h(C_{p,0}(X)) = C_{p,0}(Y)$ and $\hat{d}(h, 1) < \varepsilon$. □

In [38] it was proved that for a non-locally compact countable space X, $C_p^*(X)$ is homeomorphic to σ_ω. He did not prove theorem 3.2.13 (b) for $C_{p,0}^*(X)$. To show that in this case theorem 3.2.13 (b) is also valid we will give, for a non-locally compact countable space X, a Q-matrix $\mathcal{A}$ such that $C_{p,0}^*(X) = \ker \mathcal{A}$. Since the calculations are more or less the same as in the case of $C_{p,0}(X)$ we will be brief.

Let X be a countable space which fails to be locally compact at some point $x_0 \in X$. Again let $X = \{x_0, x_1, x_2, \ldots\}$, $Y = \{x_1, x_2, \ldots\}$, $P = \{0\} \times [-1, 1]^Y$ and let ϕ be the map $\pi_1 \circ \psi$: $\mathbb{N} \to X$. We consider the same clopen bases as above. As mentioned in the introduction, $C_{p,0}^*(X) = \{f \in C_{p,0}(X) : f(X) \subset [-1 + 1/m, 1 - 1/m]$ for some $m \in \mathbb{N}\}$.

For every $x \in X$ and $n, m \in \mathbb{N}$, define

$$B_m^{(x,n)} = \{g \in \{0\} \times [-1 + \frac{1}{m}, 1 - \frac{1}{m}]^Y : g(U_m^x) \subset [g(x) - \frac{1}{n}, g(x) + \frac{1}{n}]\}.$$

For every $n, m \in \mathbb{N}$, define $A_1^n = \emptyset$, and for $m > 1$,

$$A_m^n = B_m^{(x_0,n)} \cap \bigcap_{j=1}^{n} B_m^{(\phi(j),n)}.$$

As in lemma 3.2.4 we have the following:

3.2.14 LEMMA: $C_{p,0}^*(X) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_m^n$. □

One can prove that $\mathcal{A} = \{ A_m^n : n, m \in \mathbb{N} \}$ is a Q-matrix in P. The proof is more or less the same as the proof of theorem 3.2.12 and we will only give some remarks.

It is not necessary to copy the proof of corollary 3.2.7. We have $A_m^n \in \mathcal{I}(P)$ because of lemma 3.1.4. Furthermore we can simplify the proof of the claim in lemma 3.2.10: The condition in (5) can be skipped but we need $i_0 > i_m$ to prove $f \in A_{i_0}^{n_m+p}$. The function f can be defined as follows:

$$f(x) = \begin{cases} g(\phi(j)) & \text{if } x \in U_{i_0}^{\phi(j)} \ (j \leq n_m+p), \\ 0 & \text{if } x \in U_{i_0}^{x_0}, \\ g(x) & \text{elsewhere.} \end{cases}$$

As in theorem 3.2.13, we have

3.2.15 THEOREM: *Let X be a non-locally compact countable space. Then*

(a) $C_p^*(X)$ *is homeomorphic to* σ_ω, *and*

(b) If Y is another non-locally compact countable space, then for every $\varepsilon > 0$ *there is* $h \in \mathcal{H}(Q)$ *such that* $h(C_{p,0}^*(X)) = C_{p,0}^*(Y)$ *and* $\hat{d}(h, 1) < \varepsilon$.

The question remains whether for non-locally compact countable spaces X and Y there is a homeomorphism from the Hilbert cube onto itself arbitrary close to the identity which maps $C_p(X)$ onto $C_p(Y)$ resp. $C_p^*(X)$ onto $C_p^*(Y)$. By theorem 3.2.13, we have for $\varepsilon > 0$, $h \in \mathcal{H}([-1, 1] \times P)$ such that $\hat{d}(h, 1) < \varepsilon$, and

$$h((-1, 1) \times C_{p,0}(X)) = (-1, 1) \times C_{p,0}(Y).$$

By proposition 2.3.2, $(-1, 1) \times C_{p,0}(X)$ is homeomorphic to $C_p(X)$. This is not what we need to solve the above question. We actually need a homeomorphism from $[-1, 1] \times Q$ to Q which maps $(-1, 1) \times C_{p,0}(X)$ onto $C_p(X)$. Whether such a homeomorphism exists remains unsolved.

§3.3. Homeomorphic function spaces part 2

In this section we give another proof of the statement that for a non-locally compact countable space X the function space $C_p(X)$ is homeomorphic to σ_ω. We first compare the strategies followed in this section and section 3.2. In section 3.2 we found a Q-matrix $\mathcal{A}$ such that $\ker \mathcal{A} = C_{p,0}(X)$. Using corollary 3.1.13, it was then easily deduced

that $C_p(X)$ and σ_ω are homeomorphic. The Q-matrix involved asked for a lot of technical calculations. However this was not a waste of time, since this strategy also gives the result stated in theorem 3.2.13 (b). The strategy in this section starts with a test space T. One could say that T is the "simplest" non-locally compact countable space: moreover T is a closed subspace of any non-locally compact countable space (lemma 3.3.1). Next a Q-matrix $\mathcal{B}$ will be given such that $\ker \mathcal{B} = C_{p,0}(T)$. This Q-matrix is much easier to deal with than the one in section 3.2. It follows that $C_p(T)$ is homeomorphic to σ_ω. To get this also for arbitrary non-locally compact spaces X we use strong results of Toruńczyk [50], [51], which gives the necessary connection between $C_p(X)$ and $C_p(T)$. The method of this section was used by van Mill in [38], where he proved that $C_p^*(X)$ is homeomorphic to σ_ω.

We first define the test space T. The underlying set of T is $\mathbb{N}^2 \cup \{\infty\}$. Each point of $\mathbb{N}^2$ is isolated and $\{(\{n, n+1, \cdots\} \times \mathbb{N}) \cup \{\infty\}\}_{n \in \mathbb{N}}$ is a local open base at ∞. Then T is obviously a countable space which is not locally compact at ∞. Among the non-locally compact countable spaces, T is a special one as is shown in the following

3.3.1 LEMMA: *Let X be a non-locally compact space. Then X contains a closed copy of T*

PROOF: Let x_0 be a point where X fails to be locally compact. Let $\{U_n : n \in \mathbb{N}\}$ be a decreasing open base at x_0. Since no $\overline{U}_n$ is compact we may assume that for each $n \in \mathbb{N}$, $\overline{U}_n \setminus U_{n+1}$ contains an infinite closed discrete subset D_n. Let $S = \{x\} \cup \bigcup_{n=1}^{\infty} D_n$. Then S is obviously closed in X and homeomorphic to T. □

Recall from the introduction that $C_p(T) = \{f : T \to (-1, 1) : f \text{ is continuous}\}$ and $C_{p,0}(T) = \{f \in C_p(T) : f(\infty) = 0\}$.

For convenience let $\mathrm{I} = [-1, 1]$, $\mathrm{I}_m = [-1 + 1/m, 1 - 1/m]$ for every $m \in \mathbb{N}$ and

$$B(\varepsilon) = \prod_{i=1}^{\infty} [-\varepsilon, \varepsilon]_i \text{ for every } \varepsilon > 0.$$

For every $i \in \mathbb{N}$, let $Q_i = \Pi_{j=1}^{\infty} \mathrm{I}_{ij}$, where $\mathrm{I}_{ij} = \mathrm{I}$ for every $j \in \mathbb{N}$. Let $P = \Pi_{i=1}^{\infty} Q_i$, where $Q_i = Q$ for every $i \in \mathbb{N}$. Observe that there is a convexity preserving homeomorphism between P and Q, hence by theorem 3.1.1 each Keller space in P is a Hilbert cube. Define ϕ: $C_{p,0}(T) \to P$ by

$$\phi(f)_i = (f((i,j)))_{j \in \mathbb{N}}, \text{ for every } i \in \mathbb{N}.$$

Then ϕ is easily seen to be an embedding.

For every $n, m \in \mathbb{N}$ define $A_m^n \subset P$ by

(1) $A_1^n = \emptyset$ for every $n \in \mathbb{N}$ and

(2) $A_m^n = \prod_{i=1}^{m}((I_m)^n \times I \times I \times \cdots)_i \times \prod_{i=m+1}^{\infty} B_i(2^{-n})$ for every $n \in \mathbb{N}$ and $m \geq 2$.

It will turn out that $\mathcal{A} = \{A_m^n : n, m \in \mathbb{N}\}$ is a Q-matrix in P such that its kernel is equal to $\phi(C_{p,0}(T))$. As will be clear in the sequel the calculations involved are not so comprehensive as the ones in section 3.2.

3.3.2 LEMMA: *$\mathcal{A}$ is a $\mathcal{I}$-matrix in P.*

PROOF: By lemma 3.1.4 we have for every $n, m \in \mathbb{N}$, that $A_m^n \in \mathcal{I}(P)$. It is clear that for every $n, m \in \mathbb{N}$, $A_m^n \subset A_{m+1}^n$ and $A_m^{n+1} \subset A_m^n$. □

3.3.3 LEMMA: $\ker \mathcal{A} = \phi(C_{p,0}(T))$.

PROOF: Let $f = (f_{ij})_{(i,j)\in\mathbb{N}^2} \in \ker \mathcal{A}$ and $(i, j) \in \mathbb{N}^2$. Define $\tilde{f}: T \to (-1, 1)$ by $\tilde{f}((i, j)) = f_{ij}$ for $(i, j) \in \mathbb{N}^2$ and $\tilde{f}(\infty) = 0$. Since $f \in \bigcup_{m=1}^{\infty} A_m^j$, there is $m \in \mathbb{N}$ with $f \in A_m^j$. If $i \leq m$ then $f_{ij} \in I_m \subset (-1, 1)$ and if $i > m$ then $f_{ij} \in [-2^{-j}, 2^{-j}] \subset (-1, 1)$, hence $\tilde{f}$ is well-defined. We will prove that $\tilde{f}$ is continuous. To this end we only have to prove that $\tilde{f}$ is continuous at ∞. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Let $m \in \mathbb{N}$ be such that $f \in A_m^n$. Then $|f_{ij}| \leq 2^{-n} < \varepsilon$ for $i > m$ and $j \in \mathbb{N}$. So $\tilde{f}((\{m+1, m+2, \cdots\} \times \mathbb{N}) \cup \{\infty\}) \subset (-\varepsilon, \varepsilon)$, hence $\tilde{f} \in C_{p,0}(T)$. Obviously $\phi(\tilde{f}) = f$, so $f \in \phi(C_{p,0}(T))$.

Conversely let $f \in C_{p,0}(T)$ and $n \in \mathbb{N}$. Since f is continuous at ∞, there is $m_1 \in \mathbb{N}$ with $|f(i, j)| < 2^{-n}$ for $i > m_1$ and $j \in \mathbb{N}$. There is $m \geq m_1$ such that for every $i \leq m$ and $j \leq n$ we have $|f(i, j)| \leq 1 - 1/m$. Then $\phi(f) \in A_m^n$ and we conclude that $\phi(f) \in \ker \mathcal{A}$. □

3.3.4 LEMMA: *$\mathcal{A}$ is a Q-matrix in P.*

PROOF: By lemma 3.3.2 $\mathcal{A}$ is a $\mathcal{I}$-matrix in P. Notice that by lemma 3.1.4 we have for every $\varepsilon > 0$ and $\delta < \varepsilon$ that $B(\delta) \in \mathcal{I}(B(\varepsilon))$.

CLAIM 1: For every $n \in \mathbb{N}$, $\{A_m^n\}_{m>1}$ is a skeleton in P.

Notice that P is a Keller space. To prove this claim we verify the conditions in

theorem 3.1.7. By lemma 3.1.4 we have for every $n, m \in \mathbb{N}$ that $A^n_m \in \mathcal{I}(P)$ and $A^n_m \in \mathcal{I}(A^n_{m+1})$. Because each A^n_m $(m > 1)$ is a product of non-degenerate intervals, it is convex and infinite-dimensional. To prove that for every $n \in \mathbb{N}$, $\bigcup_{m=1}^{\infty} A^n_m$ is dense in P, let $f \in P$ and $U = \Pi_{(i,j) \in \mathbb{N}^2} U_{ij}$ a standard neighborhood of f in P. There is $m_1 \in \mathbb{N}$ such that $i > m_1$ implies $U_{ij} = \mathrm{I}$ for every $j \in \mathbb{N}$. There is $m \geq m_1$ such that $U_{ij} \cap \mathrm{I}_m \neq \emptyset$ for every $i, j \in \mathbb{N}$. We claim that $U \cap A^n_m \neq \emptyset$. Indeed, let $(i, j) \in \mathbb{N}^2$. If $i \leq m$ then $\pi_{ij}(U \cap A^n_m) \supset U_{ij} \cap \mathrm{I}_m \neq \emptyset$. If $i > m$ then $\pi_{ij}(U \cap A^n_m) = \mathrm{I} \cap B(2^{-n}) \neq \emptyset$. We conclude that $\{A^n_m\}_{m>1}$ is a skeleton in P.

Now let $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$. As mentioned in section 3.1 we may assume $i_1 < \cdots < i_m$

CLAIM 2: $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is is a Keller space in P, hence a Hilbert cube.

Since each $A^{n_k}_{i_k}$ is a product of closed intervals, $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is a product of closed intervals. Hence $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is compact and convex. Since $A^{n_m}_{i_1} \subset \bigcap_{k=1}^{m} A^{n_k}_{i_k}$ and $i_1 \neq 1$, $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is infinite-dimensional. We conclude that $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is a Keller space in P. Hence by theorem 3.1.1, $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is a Hilbert cube.

CLAIM 3: For every $p \in \mathbb{N}$, $\{\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$.

By claim 2, $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is a Keller space in P. We prove this claim by verifying the conditions in theorem 3.1.7. Let $p \in \mathbb{N}$ and $i \in \mathbb{N} \setminus \{1\}$. Let j be greater than $\max(i, i_m)$. The j-th factor space of $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$ is $B(2^{-n_m})$ and the j-th factor space of $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i$ is $B(2^{-n_m-p})$, so we have $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i \in \mathcal{I}(\bigcap_{k=1}^{m} A^{n_k}_{i_k})$ (by lemma 3.1.4).

If $i \geq i_m$, then the $(i+1)$-th factor space of $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i$ is $B(2^{-n_m-p})$ and the $(i+1)$-th factor space of $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_{i+1}$ is $B(2^{-n_m})$. Hence by lemma 3.1.4 we have for every $i \geq i_m$, $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i \in \mathcal{I}(\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_{i+1})$. Since $\bigcup_{i \geq i_m} (\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i)$ is dense in $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$, we have by theorem 3.1.7 that $\{\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i\}_{i \geq i_m}$ is a skeleton in $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$. Hence by the definition of skeletons, $\{\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^{n_m+p}_i\}_{i>1}$ is a skeleton in $\bigcap_{k=1}^{m} A^{n_k}_{i_k}$.

CLAIM 4: For every $s \in \mathbb{N}$ and for every $t \in \mathbb{N}$ such that $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \not\subset A^s_t$ we have $\bigcap_{k=1}^{m} A^{n_k}_{i_k} \cap A^s_t \in \mathcal{I}(\bigcap_{k=1}^{m} A^{n_k}_{i_k})$.

If $s > n_m$, then by claim 3, $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$. If $s \leq n_m$, there is $p \leq m$ such that $n_{p-1} < s \leq n_p$ (let $n_0 = 0$). This implies $t < i_p$, otherwise $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \subset A_{i_l}^{n_l} \subset A_t^s$. So there is $r \leq p$ such that $i_{r-1} < t+1 \leq i_r$ (let $i_0 = 0$). The $(t+1)$-th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is $B(2^{-n_{r-1}})$ and the $(t+1)$-th factor space of $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s$ is $B(2^{-s})$. Because $s > n_{p-1} \geq n_{r-1}$, by lemma 3.1.4 $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_t^s \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$.

By claims 1-4 we have that $\mathcal{A}$ is a Q-matrix in P. □

3.3.5 COROLLARY: *$C_p(T)$ is homeomorphic to σ_ω.*

PROOF: By lemmas 3.3.3, 3.3.4 and corollary 3.1.13 (a), we have that $C_{p,0}(T)$ is homeomorphic to σ_ω. As in theorem 3.2.13 we can prove that $C_p(T)$ is homeomorphic to σ_ω. □

In [38] van Mill constructed a Q-matrix $\mathcal{A}$ such that $C_{p,0}^*(T) = \ker \mathcal{A}$, hence $C_p^*(T)$ and σ_ω are homeomorphic. From this result he derived for an arbitrary non-locally compact countable space X, that $C_p^*(X)$ and σ_ω are homeomorphic. We proceed in the same way to derive that $C_p(X)$ and σ_ω are homeomorphic. We first need results of Toruńczyk (cf. theorem 3.3.6).

For a linear space E we define $\Sigma E = \{x \in E^\infty : x_i = 0 \text{ for all but finitely many } i\}$.

3.3.6 THEOREM: *Let E be a locally convex linear space. Then*

(a) ([50]) *for a closed AR X in E we have $X \times \Sigma E \approx \Sigma E$, and*

(b) ([51]) $\Sigma(E^\infty) \approx \Sigma\mathbb{R} \times E^\infty$.

This theorem will be used in the proof of theorem 3.3.9, which formulates in a sense the connection between $C_p(T)$ and $C_p(X)$ for an arbitrary non-locally compact space X. Before we come to this theorem we have to prove some lemmas. Recall from examples 3.1.8 and 3.1.14 that $\Sigma = \{x \in Q : \exists n \in \mathbb{N}\ \forall i \in \mathbb{N},\ |x_i| \leq 1 - 1/n\}$ is a skeletoid in Q and that Σ^∞ is homeomorphic to σ_ω.

3.3.7 LEMMA: *Let X be a σ-compact space. Then Σ contains a closed copy of X.*

PROOF: Since every space admits an embedding in the Hilbert cube, and $[-1/2, 1/2]^\infty$ is a Hilbert cube, X has a compactification $\alpha X \subset Q$ such that $\alpha X \in \mathcal{I}(Q)$ (cf. lemma 3.1.4). Since X is σ-compact we then have by lemma 3.1.3 (a), $X \in \mathcal{I}_\sigma(Q)$. Then by corollary 3.1.11, there is a homeomorphism $h: Q \to Q$ such that

$h(\alpha X) \cap \Sigma = h(X)$. Consequently $h(X)$ is a closed copy of X in Σ. □

A space X is called an ***absolute*** $F_{\sigma\delta}$ if X is an $F_{\sigma\delta}$ in Y, for every space Y in which X is embedded. It is well-known that a space X is an absolute $F_{\sigma\delta}$ whenever it is an $F_{\sigma\delta}$ in some completely metrizable space.

3.3.8 COROLLARY: *Let X be an absolute $F_{\sigma\delta}$. Then σ_ω contains a closed copy of X.*

PROOF: We may assume that X is a subspace of Q. Let $X = \bigcap_{i=1}^{\infty} F_i$, where each F_i is a F_σ-subspace of Q. Then each F_i is σ-compact. So by lemma 3.3.7 there exists for every $i \in \mathbb{N}$ a closed embedding $f_i: F_i \to \Sigma$. Now define $f: X \to \Sigma^\infty$ by $f(x) = (f_1(x), f_2(x), \cdots)$. It is easily seen that f is a continuous injection.

Define $\phi: X \to \Pi_{i=1}^{\infty} F_i$ by $\phi(x) = (x, x, \cdots)$ and $g: \Pi_{i=1}^{\infty} F_i \to \Sigma^\infty$ by $g((x_i)_{i\in\mathbb{N}}) = (f_i(x_i))_{i\in\mathbb{N}}$. Then $f = g \circ \phi$ and g is easily seen to be a closed embedding. Hence to prove that f is a closed embedding it suffices to prove that $\phi(X)$ is closed in $\Pi_{i=1}^{\infty} F_i$. Let $y = (y_i)_{i\in\mathbb{N}} \in \Pi_{i=1}^{\infty} F_i \backslash \phi(X)$. Then there are $i, j \in \mathbb{N}$ with $y_i \neq y_j$. There are U open in F_i and V open in F_j such that $y_i \in U$ and $y_j \in V$ and $U \cap V = \emptyset$. Let

$$O = F_1 \times \cdots F_{i-1} \times U \times F_{i+1} \cdots \times F_{j-1} \times V \times F_{j+1} \times \cdots$$

Then $y \in O$ and $O \cap \phi(X) = \emptyset$. We conclude that f is a closed embedding. Since Σ^∞ and σ_ω are homeomorphic, we are done. □

3.3.9 THEOREM: *Let X be an absolute $F_{\sigma\delta}$ which moreover is an AR. Then $X \times \sigma_\omega$ is homeomorphic to σ_ω.*

PROOF: By corollary 3.3.8, we may assume that X is closed in σ_ω. Then by theorem 3.3.6 (a), $X \times \Sigma\sigma_\omega$ and $\Sigma\sigma_\omega$ are homeomorphic. By theorem 3.3.6 (b) we have

$$\Sigma\sigma_\omega \approx \Sigma(\sigma_\omega)^\infty \approx \Sigma\mathbb{R} \times (\sigma_\omega)^\infty \approx l_f^2 \times (\sigma_\omega)^\infty \approx \sigma_\omega.$$

So we conclude $X \times \sigma_\omega$ is homeomorphic to σ_ω. □

In the proof of theorem 3.3.11 it will be clear how this theorem connects $C_p(T)$ with $C_p(X)$, for arbitrary non-locally compact countable spaces X. We need one more lemma.

3.3.10 LEMMA ([19]): *If X is a countable space, then $C_p(X)$ is an absolute $F_{\sigma\delta}$.*

PROOF: For every $x \in X$, let $\{U_n^x\}_{n\in\mathbb{N}}$ be a decreasing clopen base at x. Then by the definition of continuity,

$$C_p(X) = \bigcap_{x\in X}\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\{g \in \mathbb{R}^X : g(U_m^x) \subset [g(x) - \tfrac{1}{n}, g(x) + \tfrac{1}{n}]\}.$$

Since each set $\{g \in \mathbb{R}^X : g(U_m^x) \subset [g(x) - \frac{1}{n}, g(x) + \frac{1}{n}]\}$ is closed in $\mathbb{R}^X$, $C_p(X)$ is an $F_{\sigma\delta}$ in $\mathbb{R}^X$ and hence an absolute $F_{\sigma\delta}$. □

3.3.11 THEOREM: *Let X be a non-locally compact countable space. Then $C_p(X)$ is homeomorphic to σ_ω.*

PROOF: By lemma 3.3.1 we may assume that T is a closed subspace of X. Then by proposition 2.3.2, $C_p(X)$ and $C_{p,T}(X)\times C_p(T)$ are homeomorphic. Since $C_{p,T}(X)$ is a linear subspace of the locally convex space $C_p(X)$ it is locally convex as well. By The Dugundji Extension Theorem [39, Th. 1.4.13], $C_{p,T}(X)$ is an AR. It is easily seen that $C_{p,T}(X)$ is closed in $C_p(X)$, hence by lemma 3.3.10, $C_{p,T}(X)$ is an absolute $F_{\sigma\delta}$. So by theorem 3.3.9, $C_{p,T}(X)\times\sigma_\omega$ and σ_ω are homeomorphic. We conclude that by corollary 3.3.5

$$C_p(X) \approx C_{p,T}(X)\times\sigma_\omega \approx \sigma_\omega. \square$$

Theorem 3.3.11 can be found in [6].

§3.4. Remarks

Van Mill conjectured in [38] that for a non-discrete countable space X, $C_p^*(X)$ and σ_ω are homeomorphic. In the preceding sections, it became clear that Q-matrices were a handy tool to prove for non-locally compact countable spaces X, that $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ_ω. The question remains whether this also holds for arbitrary non-discrete countable spaces. Recently Dobrowolski, Gulko and Mogilski in [20] and Cauty in [16] independently answered this question in the affirmative. In this section we shortly discuss both papers.

In section 3.3 the test space T plays an important role. It is the "simplest" non-locally compact countable space, which is a closed subspace of any non-locally compact countable space. In the class of non-discrete countable spaces, the role of T is

played by $[1, \omega]$ because any non-discrete countable space contains a closed copy of $[1, \omega]$. Following the strategy in section 3.3, we have to prove that $C_p([1, \omega])$ is homeomorphic to σ_ω, to obtain for arbitrary non-discrete countable spaces X, that $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ_ω. Both in [16] and [20] it is proved that $C_{p,0}([1, \omega]) = \{f \in C_p([1, \omega]) : f(\omega) = 0\}$ is homeomorphic to σ_ω (hence $C_p([1, \omega])$ is homeomorphic to σ_ω). The approaches in both papers are in a sense the same: they both rely on theorems of Bestvina and Mogilski [13].

First we discuss the proof in [20]. For a space X and $x \in X$, let

$$W(X, x) = \{x \in X^\infty : x_n = x \text{ for all but finitely many } n\}.$$

We have the following characterization of σ_ω,

3.4.1 THEOREM ([20]): *An AR X is homeomorphic to σ_ω iff the following conditions are satisfied,*

(a) $X = \bigcup_{j=1}^{\infty} X_j$, where each X_j is an absolute $F_{\sigma\delta}$ and a Z-set in X,

(b) there is $x \in X$ and there is a copy Y of X such that $W(X, x) \subset Y \subset X^\infty$ and,

(c) X contains a closed copy of X^∞.

The proof of this theorem depends strongly on results derived by Bestvina and Mogilski in [13]. In [20] it is proved that $C_{p,0}([1, \omega])$ satisfies the conditions in theorem 3.4.1, so that $C_{p,0}([1, \omega])$ is homeomorphic to σ_ω. Hence following the strategy of section 3.3, for non-discrete countable spaces X, $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ_ω.

The proof in [16] depends on a theorem derived by Bestvina and Mogilski in [13]. Before we can formulate this theorem we have to give some definitions.

Let X and Y be spaces and let $\mathcal{U}$ be an open cover of Y. Two functions $f, g \in C(X, Y)$ are said to be ***$\mathcal{U}$-close*** if for every $x \in X$, there is $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. We have to extend the definition of a Z-set to arbitrary spaces. A closed subspace A of X is called a ***Z-set*** in X, whenever for every open cover $\mathcal{U}$ of X and for every $f \in C(Q, X)$, there is $g \in C(Q, X)$ $\mathcal{U}$-close to f and $g(Q) \cap A = \emptyset$. For compact spaces, this definition coincides with the one given in section 3.1. For an ANR X, we have by [39, Th. 7.2.5]:

> A closed subset A of X is a Z-set in X iff for every open cover $\mathcal{U}$ of X there exists $f \in C(X, X)$ such that f and 1_X are $\mathcal{U}$-close and $f(X) \cap A = \emptyset$.

For an ANR X, a closed subset A of X is said to be a ***strong Z-set*** if for every open cover $\mathcal{U}$ of X there exists a continuous function $f: X \to X$ such that f and 1_X are $\mathcal{U}$-close and $\overline{f(X)} \cap A = \emptyset$. A Z-set need not to be a strong Z-set (an example is given in [12]).

A space X is ***strongly $\mathcal{F}_{\sigma\delta}$-universal*** if for every $f \in C(A, X)$, where A is an absolute $\mathcal{F}_{\sigma\delta}$, for every $B \subset A$ closed such that $f \mid B: B \to X$ is a Z-embedding, and for every open cover $\mathcal{U}$ of X, there exists a Z-embedding $h: A \to X$ such that $h \mid B = f \mid B$ and f and h are $\mathcal{U}$-close.

We can now state the announced theorem of Bestvina and Mogilski,

3.4.2 THEOREM ([13]): *An AR X which is an absolute $F_{\sigma\delta}$ is homeomorphic to σ_ω iff the following conditions are satisfied*

(a) X is strongly $F_{\sigma\delta}$-universal, and

(b) $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is a strong Z-set in X.

In [16] it is proved that $C_{p,0}([1, \omega])$ satisfies the conditions of theorem 3.4.2 and hence $C_{p,0}([1, \omega])$ is homeomorphic to σ_ω. So again we have that for non-discrete countable spaces X, $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ_ω.

Let $X = \{x_0, x_1, x_2, \cdots\}$ be a countable space which is not discrete at x_0. Now that we have that $C_p(X)$ and $C_p^*(X)$ are homeomorphic to σ_ω, the question remains whether theorem 3.2.13 (b) also holds for this X. That is if Y is another non-discrete countable space, is it true then that there is $h \in \mathcal{H}(Q)$ arbitrary close to the identity which maps $C_{p,0}(X)$ onto $C_{p,0}(Y)$. For this purpose we actually would like to write $C_{p,0}(X)$ as the kernel of a Q-matrix. The Q-matrix in section 3.2 essentially uses the non-locally compactness of X, and as far as we see it cannot be used for non-discrete countable spaces. A weaker question is whether $C_{p,0}([1, \omega])$ can be written as the kernel of a Q-matrix. As with the test space T, there is a natural candidate. However this candidate unfortunately is not a Q-matrix. We shall present it and prove that it is not a Q-matrix.

We identify $C_{p,0}([1, \omega])$ with the following subspace of Q,

$$\{(x_n)_{n \in \mathbb{N}} \in s : \lim_{n \to \infty} x_n = 0\}.$$

For convenience for every $n, m \in \mathbb{N}$, let

$$I_m = [-1 + \frac{1}{m}, 1 - \frac{1}{m}] \text{ and } I_{mn} = [-(1 - \frac{1}{m})2^{-n}, (1 - \frac{1}{m})2^{-n}].$$

For every $n, m \in \mathbb{N}$, let

(1) $A_1^n = \emptyset$, and

(2) $A_m^n = (\mathrm{I}_m)^m \times \mathrm{I}_{mn} \times \mathrm{I}_{mn} \times \mathrm{I}_{mn} \times \cdots$, if $m \geq 2$.

Let $\mathcal{A} = \{ A_m^n : n, m \in \mathbb{N} \}$. By lemma 3.1.4 it is easily seen that $\mathcal{A}$ is a $\mathcal{I}$-matrix in Q. Again by lemma 3.1.4 we have for every n, $m \in \mathbb{N}$ that $A_m^n \in \mathcal{I}(A_{m+1}^n)$. Since each A_m^n is convex and infinite-dimensional and since $\bigcup_{m \in \mathbb{N}} A_m^n$ is dense in Q, we have by theorem 3.1.7, that $\{A_m^n\}_{m>1}$ is a skeleton in Q.

Now fix $n_1 < \cdots < n_m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{N} \setminus \{1\}$. By the observation in section 3.1 we assume $i_1 < \cdots < i_m$. Then it is easily seen that

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} = B_1 \times \cdots \times B_{i_m} \times \mathrm{I}_{i_m n_m} \times \mathrm{I}_{i_m n_m} \times \mathrm{I}_{i_m n_m} \times \cdots$$

where each B_i is a non-degenerate closed subinterval of I_m. So $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$ is a Keller space, hence a Hilbert cube. Furthermore for $p \in \mathbb{N}$ and $i \geq i_m$,

$$\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} = B_1 \times \cdots \times B_{i_m} \times (\mathrm{I}_{i_m n_m})^{i-i_m} \times \mathrm{I}_{i n_m+p} \times \mathrm{I}_{i n_m+p} \times \mathrm{I}_{i n_m+p} \times \cdots$$

By lemma 3.1.4 we have $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k})$ and $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p} \in \mathcal{I}(\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_{i+1}^{n_m+p})$. Since $\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}$ is convex and infinite-dimensional, and $\bigcup_{i=1}^{\infty} (\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p})$ is dense in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$, we have by theorem 3.1.7 that $\{\bigcap_{k=1}^{m} A_{i_k}^{n_k} \cap A_i^{n_m+p}\}$ is a skeleton in $\bigcap_{k=1}^{m} A_{i_k}^{n_k}$.

It seems that we are on the right way to prove that $\mathcal{A}$ is a Q-matrix. Unfortunately condition (d) in the definition of a Q-matrix is not satisfied. Indeed for

$$A_2^1 = [-\tfrac{1}{2}, \tfrac{1}{2}] \times [-\tfrac{1}{2}, \tfrac{1}{2}] \times [-\tfrac{1}{4}, \tfrac{1}{4}] \times [-\tfrac{1}{4}, \tfrac{1}{4}] \times [-\tfrac{1}{4}, \tfrac{1}{4}] \times \cdots$$

and

$$A_3^2 = [-\tfrac{1}{3}, \tfrac{1}{3}] \times [-\tfrac{1}{3}, \tfrac{1}{3}] \times [-\tfrac{1}{3}, \tfrac{1}{3}] \times [-\tfrac{1}{6}, \tfrac{1}{6}] \times [-\tfrac{1}{6}, \tfrac{1}{6}] \times \cdots$$

we have

$$A_2^1 \cap A_3^2 = [-\tfrac{1}{3}, \tfrac{1}{3}] \times [-\tfrac{1}{3}, \tfrac{1}{3}] \times [-\tfrac{1}{4}, \tfrac{1}{4}] \times [-\tfrac{1}{6}, \tfrac{1}{6}] \times [-\tfrac{1}{6}, \tfrac{1}{6}] \times \cdots$$

Furthermore $A_3^2 \not\subset A_2^1$. However by lemma 3.1.3 (b), $A_2^1 \cap A_3^2 \notin \mathcal{I}(A_3^2)$. So this natural description of $C_{p,0}([1, \omega])$ fails to be a Q-matrix, hence the question remains open whether $C_{p,0}([1, \omega])$ can be described as the kernel of a Q-matrix.

Our last remark in this section concerns uniform spaces. At this moment there ex-

ists a topological classification of function spaces $C_p(X)$, for countable non-discrete spaces X and an isomorphical classification of function spaces $C_p(X)$, for countable infinite compact spaces X (cf. chapter 2). We can also consider $C_p(X)$ as a uniform space. The uniformity on $C_p(X)$ is given by the subbase $\{U(P, \varepsilon) : P \subset X \text{ finite}, \varepsilon > 0\}$, where

$$U(P, \varepsilon) = \{(f, g) \in C_p(X) \times C_p(X) : |f(x) - g(x)| < \varepsilon \text{ for every } x \in P\}.$$

Since every linear homeomorphism is a uniform homeomorphism and every uniform homeomorphism is a homeomorphism, it is interesting to find a uniform classification of the function spaces $C_p(X)$, for countable infinite compact spaces X. In [27], Gulko derived the following

3.4.3 THEOREM ([27]): *Let X be a countable infinite compact space. Then $C_p(X)$ is uniformly homeomorphic to $C_p([1, \omega])$.*

So for countable infinite compact spaces the topological and uniform classification coincide. As a corollary we also have that there are spaces X and Y such that $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic but not linearly homeomorphic. In [27] Gulko announces a complete uniform classification of $C_p(X)$ for all countable metric spaces.

CHAPTER 4

On the ℓ_p-equivalence of metric spaces

All spaces considered in this chapter are Tychonov.

In chapter 3 we stated a topological classification result for the spaces $C_p(X)$, where X is any countable metric space. In the light of this result the question naturally arises which of these function spaces are in fact linearly homeomorphic, i.e., isomorphic as linear spaces. In chapter 2, we already obtained an isomorphical classification of the spaces $C_p(X)$, where X is any locally compact zero-dimensional separable metric space. In this chapter we also consider non-locally compact zero-dimensional separable metric spaces.

In section 4.1, we introduce the notion of ℓ_p-equivalent pairs, which is a useful tool in deriving topological properties of metric spaces which are preserved by ℓ_p-equivalence. In section 4.2 we show that the topological properties preserved by ℓ_p-equivalence, found in section 4.1, are sufficient to give an isomorphical classification of the function spaces $C_p(X)$, where X is any countable metric space with scattered height less than or equal to ω. Unfortunately these properties are not sufficient to give a complete isomorphical classification for the class of all countable metric spaces X. The results in sections 4.1 and 4.2 can be found in [4]. In section 4.3, we present other topological properties preserved by ℓ_p-equivalence. In section 4.4, we state a conjecture on a complete isomorphical classification for the function spaces considered. Some remarks are made concerning the difficulties one encounters when one attempts to prove the conjecture. Finally, some partial results are given on ℓ_0-equivalence (section 4.5) and ℓ_p^*-equivalence (section 4.6).

§4.1. ℓ_p-equivalent properties of metric spaces

In this section we present ℓ_p-equivalent properties of metric spaces. The notion of

ℓ_p-equivalent pairs provides us with these properties. Before we give the definition of ℓ_p-equivalent pairs, we first need the following:

Let X be a space and $X_0 \subset X$. For every ordinal we define the set $X^{\{\alpha\}}$ with respect to the pair (X, X_0) by transfinite induction as follows:

(1) $X^{\{0\}} = X_0$.

(2) If α is a successor, say $\alpha = \beta + 1$, then $x \in X^{\{\alpha\}}$ if and only if for every neighborhood U of x, $\overline{U} \cap X^{\{\beta\}}$ is not compact.

(3) If α is a limit ordinal, then $X^{\{\alpha\}} = \bigcap_{\beta<\alpha} X^{\{\beta\}}$.

The construction of the sets $X^{\{\alpha\}}$ is a special case of a construction in [49]. Note that, whereas taking the derivative of a space means "throwing away all isolated points", the above procedure throws away all points with a compact neighborhood. There are also some similarities between both operations which are formulated in the following two lemmas. They will be used frequently but will not always be mentioned. For a subset U of X, we define as above for every ordinal α, the set $U^{\{\alpha\}}$ with respect to the pair (U, U_0), where $U_0 = U \cap X_0$. Compare the following two lemmas with proposition 2.2.2, corollary 2.2.3 and proposition 2.2.4.

4.1.1 LEMMA: *Let X be a space and X_0 a closed subspace of X. Then for every ordinal α,*

(a) $X^{\{\alpha\}}$ is closed in X,

(b) $X^{\{\alpha\}} \subset X^{(\alpha)}$,

(c) for $\beta < \alpha$, $X^{\{\alpha\}} \subset X^{\{\beta\}}$, and

(d) $X^{\{\alpha+1\}} = (X^{\{\alpha\}})^{\{1\}}$.

PROOF: We prove each case by transfinite induction on α.

For (a), the case $\alpha = 0$ is a triviality. First suppose that $\alpha > 0$ is a successor, say $\alpha = \beta + 1$. Let $x \in X \setminus X^{\{\alpha\}}$. Then there is an open neighborhood U of x such that $\overline{U} \cap X^{\{\beta\}}$ is compact. So $U \cap X^{\{\alpha\}} = \emptyset$, hence $X^{\{\alpha\}}$ is closed. Secondly, if α is a limit ordinal, then $X^{\{\alpha\}} = \bigcap_{\beta<\alpha} X^{\{\beta\}}$, so by our inductive hypothesis, $X^{\{\alpha\}}$ is closed in X. For (b), the case $\alpha = 0$ is a triviality. If $\alpha > 0$ is a successor, say $\alpha = \beta + 1$, then for no neighborhood U of a point $x \in X^{\{\alpha\}}$, $\overline{U} \cap X^{\{\beta\}}$ is compact. By the inductive hypothesis we have $\overline{U} \cap X^{(\beta)}$ is not compact, hence $\overline{U} \cap X^{(\beta)} \setminus \{x\} \neq \emptyset$. We conclude that $x \in X^{(\alpha)}$. For α a limit ordinal, part (b) is clear.

For (c), first let $\alpha = 1$. If $x \notin X_0$, there is a neighborhood U of x such that $\overline{U} \cap X_0 = \emptyset$, hence $x \notin X^{\{1\}}$. So $X^{\{1\}} \subset X^{\{0\}}$. If $\alpha > 1$ is a successor, say $\alpha = \gamma + 1$, then for every

$x \in X^{\{\alpha\}}$ and for every neighborhood U of x, $\overline{U} \cap X^{\{\gamma\}}$ is not compact. Hence $\overline{U} \cap X^{\{\beta\}}$ is not compact and so $x \in X^{\{\beta\}}$. For α a limit ordinal part (c) is a triviality.

For (d), let $x \in X^{\{\alpha+1\}}$. Then for each neighborhood U of x, $\overline{U} \cap X^{\{\alpha\}}$ is not compact if and only if $\overline{U} \cap X^{\{\alpha\}} \cap X_0$ is not compact if and only if $x \in (X^{\{\alpha\}})^{\{1\}}$. □

One can now easily see that if X_0 and Y_0 are closed in X and $Y_0 \subset X_0$, then for each ordinal α the set $X^{\{\alpha\}}$ with respect to the pair (X, Y_0) is a subset of the set $X^{\{\alpha\}}$ with respect to the pair (X, X_0).

4.1.2 LEMMA: *Let X be a space and U a subset of X, and X_0 a closed subset of X. Then for each ordinal α,*

(a) if U is closed, then $U^{\{\alpha\}} \subset U \cap X^{\{\alpha\}}$, and

(b) if U is open, then $U \cap X^{\{\alpha\}} \subset U^{\{\alpha\}}$.

PROOF: We prove this proposition by transfinite induction on α. If $\alpha = 0$, the lemma is obviously true, so suppose that $\alpha > 0$ and that for each $\beta < \alpha$ the lemma has been proved. First suppose that α is a successor, say $\alpha = \beta + 1$.

For (a), suppose U is closed, let $x \in U^{\{\alpha\}}$ and let V be a neighborhood of x. Then by the inductive hypothesis

$$\overline{V} \cap U^{\{\beta\}} \subset \overline{V} \cap (U \cap X^{\{\beta\}}) \subset \overline{V} \cap X^{\{\beta\}}.$$

By lemma 4.1.1, $\overline{V} \cap U^{\{\beta\}}$ is a closed subset of U and because U is closed in X, $\overline{V} \cap U^{\{\beta\}}$ is a closed subset of $\overline{V} \cap X^{\{\beta\}}$. Since $\overline{V} \cap U^{\{\beta\}}$ is not compact, we then have that $\overline{V} \cap X^{\{\beta\}}$ is not compact. So $x \in X^{\{\alpha\}} \cap U$.

For (b), suppose U is open and let $x \in U \cap X^{\{\alpha\}}$. Let V be a neighborhood of x in X such that $\overline{V} \subset U$. Then $\overline{V} \cap X^{\{\beta\}}$ is not compact. So by the inductive hypothesis

$$\overline{V} \cap X^{\{\beta\}} = \overline{V} \cap U \cap X^{\{\beta\}} \subset \overline{V} \cap U^{\{\beta\}},$$

hence $\overline{V} \cap U^{\{\beta\}}$ is not compact. We conclude that $x \in U^{\{\alpha\}}$.

If α is a limit ordinal, then by the inductive hypothesis we have for closed U that

$$U^{\{\alpha\}} = \bigcap_{\beta<\alpha} U^{\{\beta\}} \subset \bigcap_{\beta<\alpha} (U \cap X^{\{\beta\}}) = U \cap \bigcap_{\beta<\alpha} X^{\{\beta\}} = U \cap X^{\{\alpha\}},$$

and for open U

$$U \cap X^{\{\alpha\}} = U \cap \bigcap_{\beta<\alpha} X^{\{\beta\}} = \bigcap_{\beta<\alpha} (U \cap X^{\{\beta\}}) \subset \bigcap_{\beta<\alpha} U^{\{\beta\}} = U^{\{\alpha\}}.$$

This completes the proof of the lemma. □

This lemma implies that whenever U is a clopen subset of a space X, then for each ordinal α, $U \cap X^{\{\alpha\}} = U^{\{\alpha\}}$; furthermore by proposition 2.2.4, $U^{(\alpha)} = U \cap X^{(\alpha)}$. We will use this frequently without explicit reference.

4.1.3 LEMMA: *Let X be a paracompact space, X_0 closed in X and $\alpha \geq 1$ an ordinal. Let $V \subset X$ be open such that $\overline{V} \cap X^{\{\alpha\}} = \emptyset$. Then there is a locally finite family $\{V_s : s \in S\}$ consisting of open sets such that $V = \bigcup_{s \in S} V_s$ and for every $s \in S$, there is $\beta < \alpha$ with $\overline{V}_s \cap X^{\{\beta\}}$ compact.*

PROOF: *Case 1*: α is a successor, say $\alpha = \beta + 1$.

Since $\overline{V} \cap X^{\{\alpha\}} = \emptyset$, for every $x \in \overline{V}$, there is a neighborhood U_x of x such that $\overline{U}_x \cap X^{\{\beta\}}$ is compact. Since $\{U_x : x \in \overline{V}\} \cup \{X \setminus \overline{V}\}$ is an open cover of X, there is a locally finite open refinement $\{O_s : s \in S\}$ of it. For every $s \in S$, let $V_s = O_s \cap V$. Then $\{V_s : s \in S\}$ is a locally finite family consisting of open sets such that $V = \bigcup_{s \in S} V_s$. In addition, if $s \in S$ and $V_s \neq \emptyset$ there is $x \in \overline{V}$ with $V_s \subset U_x$. Then $\overline{V}_s \cap X^{\{\beta\}} \subset \overline{U}_x \cap X^{\{\beta\}}$. So $\overline{V}_s \cap X^{\{\beta\}}$ is compact.

Case 2: α is a limit ordinal.

Then $\mathcal{U} = \{X \setminus X^{\{\beta\}} : \beta < \alpha\} \cup \{X \setminus \overline{V}\}$ is an open cover of X, so there is a locally finite open covering $\{O_s : s \in S\}$ of X such that $\{\overline{O}_s : s \in S\}$ refines $\mathcal{U}$. For every $s \in S$ put $V_s = V \cap O_s$. Then $\{V_s : s \in S\}$ is a locally finite family of open sets such that $V = \bigcup_{s \in S} V_s$. Now fix $s \in S$ and suppose $V_s \neq \emptyset$. Then there is $\beta < \alpha$ such that $\overline{V}_s \subset X \setminus X^{\{\beta\}}$, which implies $\overline{V}_s \cap X^{\{\beta\}} = \emptyset$. □

Let X be a space. There are several possibilities to combine the two operations $X^{(\alpha)}$ and $X^{\{\alpha\}}$. The one that is important for our purposes is the case where $X_0 = X^{(\alpha)}$ for some ordinal α. In the sequel the set $X^{\{\beta\}}$ with respect to $X_0 = X^{(\alpha)}$ will be denoted by $X^{(\alpha, \beta)}$. Another subset of X we need in section 4.3 is $X^{\ll \alpha, \beta \gg}$ defined for limit ordinals α by

$$X^{\ll \alpha, \beta \gg} = \bigcap_{\gamma < \alpha} X^{(\gamma, \beta)}.$$

Note that the sets $X^{(\alpha, \beta)}$ and $X^{\ll \alpha, \beta \gg}$ are closed in X, and if $\beta = 0$, then $X^{\ll \alpha, \beta \gg} = X^{(\alpha)}$. As in lemma 4.1.2 one can prove that for U clopen in X, $U \cap X^{(\alpha, \beta)} = U^{(\alpha, \beta)}$, and $U \cap X^{\ll \alpha, \beta \gg} = U^{\ll \alpha, \beta \gg}$.

In this chapter it will be made clear that the isomorphical classification of $C_p(X)$ for countable metric X depends upon the behaviour of X with respect to the above operations and that neither of the operations is redundant. We need the following lemma in

this section and also in section 4.3.

4.1.4 LEMMA: *Let X be a zero-dimensional separable metric space. Let $\alpha \geq 1$ and β be ordinals and let V be an open subset of X. Suppose that*

(a) $\overline{V} \cap X^{(\alpha)} = \emptyset$, *or*

(b) $\overline{V} \cap X^{(\beta,\alpha)} = \emptyset$, *or*

(c) α *is a limit ordinal and* $\overline{V} \cap X^{\ll\alpha,\beta\gg} = \emptyset$.

Then there is a discrete clopen family $\{A_i : i \in \mathbb{N}\}$ such that $V \subset \bigcup_{i\in\mathbb{N}} A_i$ and for each $i \in \mathbb{N}$, there is $\gamma < \alpha$ such that

if (a) holds, then $A_i^{(\gamma)}$ is finite and if moreover α is a limit, then $A_i^{(\gamma)} = \emptyset$;

if (b) holds, then $A_i^{(\beta,\gamma)}$ is compact and if moreover α is a limit, then $A_i^{(\beta,\gamma)} = \emptyset$;

if (c) holds, then $A_i^{(\gamma,\beta)} = \emptyset$.

PROOF: The proof is almost the same as the proof of lemma 4.1.3, hence we will be brief and present a proof of case (a) only.

If $\alpha = \gamma + 1$ is a successor, there is for each $x \in \overline{V}$ a clopen neighborhood U_x of x such that $U_x^{(\gamma)}$ is finite. The open cover $\{U_x : x \in \overline{V}\} \cup X \setminus \overline{V}$ of X has a clopen disjoint refinement $\{A_i : i \in \mathbb{N}\}$. Put $I = \{i \in \mathbb{N} : A_i \cap V \neq \emptyset\}$. Then $\{A_i : i \in I\}$ is a discrete clopen family which is as required.

If α is a limit ordinal, then $\mathcal{U} = \{X \setminus X^{(\beta)} : \beta < \alpha\} \cup \{X \setminus \overline{V}\}$ is an open cover of X. Let $\{A_i : i \in \mathbb{N}\}$ be a disjoint clopen refinement of $\mathcal{U}$, and put $I = \{i \in \mathbb{N} : A_i \cap V \neq \emptyset\}$. Then $\{A_i : i \in I\}$ is a discrete clopen family which satisfies the desired conditions. □

4.1.5 COROLLARY: *Let X be a zero-dimensional separable metric space. Let $\alpha \geq 1$ and β be ordinals and let V be an clopen subset of X. Suppose that*

(a) $V^{(\alpha)} = \emptyset$, *or*

(b) $V^{(\beta,\alpha)} = \emptyset$, *or*

(c) α *is a limit ordinal and* $V^{\ll\alpha,\beta\gg} = \emptyset$.

Then there is a discrete clopen family $\{A_i : i \in \mathbb{N}\}$ such that $V = \bigcup_{i\in\mathbb{N}} A_i$ and for each $i \in \mathbb{N}$, there is $\gamma < \alpha$ such that

if (a) holds, then $A_i^{(\gamma)}$ is finite and if moreover α is a limit, then $A_i^{(\gamma)} = \emptyset$;

if (b) holds, then $A_i^{(\beta,\gamma)}$ is compact and if moreover α is a limit, then $A_i^{(\beta,\gamma)} = \emptyset$;

if (c) holds, then $A_i^{(\gamma,\beta)} = \emptyset$. □

We now define some additional notions. Let X and Y be spaces. Let X_0 be closed in X and Y_0 be closed in Y. Let $\phi: C_p(X) \to C_p(Y)$ be a linear bijection and α an ordinal. We define the pair (X, X_0) to be (ϕ, α)-***relative*** to the pair (Y, Y_0) if the following

holds:

> If U and V are open in X and W is open in Y such that $(\operatorname{supp} U) \cap W = \emptyset$ and $\operatorname{supp} W \subset U \cup V$, then $W \cap Y^{\{\alpha\}} \neq \emptyset$ implies $\overline{V} \cap X^{\{\alpha\}} \neq \emptyset$.

We define (X, X_0) and (Y, Y_0) to be ***ℓ_p-equivalent pairs*** if there is a linear homeomorphism $\phi: C_p(X) \to C_p(Y)$ such that (X, X_0) is $(\phi, 0)$-relative to (Y, Y_0) and (Y, Y_0) is $(\phi^{-1}, 0)$-relative to (X, X_0). Note that two spaces X and Y are ℓ_p-equivalent if and only if $(X, \emptyset)$ and $(Y, \emptyset)$ are ℓ_p-equivalent pairs.

The importance of ℓ_p-equivalent pairs will become clear in proposition 4.1.9 and proposition 4.1.12.

4.1.6 LEMMA: *Let X and Y be metric spaces, X_0 closed in X and Y_0 closed in Y. Let $\phi: C_p(X) \to C_p(Y)$ be a continuous linear bijection such that (X, X_0) is $(\phi, 0)$-relative to (Y, Y_0). Then for every ordinal α, (X, X_0) is (ϕ, α)-relative to (Y, Y_0).*

PROOF: We prove the lemma by transfinite induction on α. Since (X, X_0) is $(\phi, 0)$-relative to (Y, Y_0), the case $\alpha = 0$ is clear. So assume the lemma to be true for every ordinal $\beta < \alpha$ with $\alpha \geq 1$. Suppose that the lemma is false for α. Then there are U and V open in X and W open in Y such that $(\operatorname{supp} U) \cap W = \emptyset$, $\operatorname{supp} W \subset U \cup V$, $W \cap Y^{\{\alpha\}} \neq \emptyset$ and $\overline{V} \cap X^{\{\alpha\}} = \emptyset$. By lemma 4.1.3, there is a locally finite family $\{V_s : s \in S\}$ consisting of open sets such that $V = \bigcup_{s \in S} V_s$ and for every $s \in S$ there is $\beta < \alpha$ such that $\overline{V}_s \cap X^{\{\beta\}}$ is compact. Choose $y \in W \cap Y^{\{\alpha\}}$ and a neighborhoodbase $\{W_m : m \in \mathbb{N}\}$ at y in W such that for every $m \in \mathbb{N}$, $\overline{W}_{m+1} \subset W_m$. By corollary 1.2.15 (a) and lemma 1.2.10, there are $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in S$ with

$$\operatorname{supp} W_m \cap \bigcup_{s \notin \{s_1, \ldots, s_m\}} V_s = \emptyset. \tag{1}$$

Now let $A = \bigcup_{i=1}^{m} V_{s_i}$. Fix $\beta < \alpha$ such that $\overline{A} \cap X^{\{\beta\}}$ is compact. Also, notice the following: A and U are open in X, W_m is open in Y, $(\operatorname{supp} U) \cap W_m = \emptyset$ (because $W_m \subset W$ and $(\operatorname{supp} U) \cap W = \emptyset$) and $\operatorname{supp} W_m \subset U \cup A$ (by (1) and the fact that $\operatorname{supp} W \subset U \cup V$). Since $y \in W_m \cap Y^{\{\beta\}}$, our inductive hypothesis implies that $\overline{A} \cap X^{\{\beta\}} \neq \emptyset$. We have that X is a metric space, so there is an open neighborhoodbase $\{A_s : s \in \mathbb{N}\}$ at $\overline{A} \cap X^{\{\beta\}}$ in X such that $\overline{A}_{s+1} \subset A_s$ for every $s \in \mathbb{N}$. Since $y \in Y^{\{\alpha\}}$ and $\overline{W}_{m+1}$ is a neighborhood of y, $\overline{W}_{m+1} \cap Y^{\{\beta\}}$ is not compact, so in Y there is a closed discrete subset $\{y_s : s \in \mathbb{N}\}$ contained in $W_m \cap Y^{\{\beta\}}$. Let $\{O_s : s \in \mathbb{N}\}$ be an open discrete family in W_m such that $y_s \in O_s$. Then by corollary 1.2.15 (a) and lemma 1.2.10,

there is $s \in \mathbb{N}$ with

$$\operatorname{supp} A_s \cap \bigcup_{i \geq s} O_i = \emptyset. \tag{2}$$

Now put $U' = U \cup A_s$, $V' = A \setminus \bar{A}_{s+1}$ and $W' = O_s$. Then U' and V' are open in X and W' is open in Y. We also have

$$(\operatorname{supp} U') \cap W' = (\operatorname{supp} U \cup \operatorname{supp} A_s) \cap O_s = \emptyset \quad \text{(by 2)}$$

and

$$\operatorname{supp} W' \subset \operatorname{supp} W_m \subset U \cup A \subset U' \cup V'.$$

Furthermore, $y_s \in W' \cap Y^{\{\beta\}}$ and

$$\overline{V'} \cap X^{\{\beta\}} = \overline{(A \setminus \bar{A}_{s+1})} \cap X^{\{\beta\}} \subset (\bar{A} \setminus A_{s+1}) \cap X^{\{\beta\}} = \emptyset.$$

This contradicts our inductive assumption. □

4.1.7 THEOREM: *Let X and Y be metric spaces, X_0 closed in X and Y_0 closed in Y. Suppose that (X, X_0) and (Y, Y_0) are l_p-equivalent pairs. Then for every ordinal α we have*

(a) $X^{\{\alpha\}} = \emptyset$ if and only if $Y^{\{\alpha\}} = \emptyset$,

(b) $X^{\{\alpha\}}$ is compact if and only if $Y^{\{\alpha\}}$ is compact, and

(c) $X^{\{\alpha\}}$ is locally compact if and only if $Y^{\{\alpha\}}$ is locally compact.

PROOF: Let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism such that (X, X_0) is $(\phi, 0)$-relative to (Y, Y_0) and (Y, Y_0) is $(\phi^{-1}, 0)$-relative to (X, X_0). For (a), by applying lemma 4.1.6 and the definition of (ϕ, α)-relativeness to $U = \emptyset$, $V = X$ and $W = Y$, we get $X^{\{\alpha\}} = \emptyset$ if $Y^{\{\alpha\}} = \emptyset$.

For (b) suppose that $Y^{\{\alpha\}}$ is compact and $X^{\{\alpha\}}$ is not. Since $X^{\{\alpha\}} \neq \emptyset$, by (a) we have $Y^{\{\alpha\}} \neq \emptyset$. Let $\{W_m : m \in \mathbb{N}\}$ be an open decreasing base in Y at $Y^{\{\alpha\}}$ such that for every $m \in \mathbb{N}$, $\overline{W}_{m+1} \subset W_m$. Furthermore, let $\{x_m : m \in \mathbb{N}\}$ be closed and discrete in $X^{\{\alpha\}}$. Let $\{O_m : m \in \mathbb{N}\}$ be an open discrete family in X such that $x_m \in O_m$. Then by corollary 1.2.15 (a) and lemma 1.2.10, there is $m \in \mathbb{N}$ such that

$$\operatorname{supp} W_m \cap \bigcup_{i \geq m} O_i = \emptyset.$$

Now let $U = W_m$, $V = Y \setminus \overline{W}_{m+1}$ and $W = O_m$. Then U and V are open, W is open, $(\operatorname{supp} U) \cap W = \emptyset$ and $\operatorname{supp} W \subset Y = U \cup V$. In addition

$$\overline{V} \cap Y^{\{\alpha\}} = \overline{Y \setminus \overline{W}_{m+1}} \cap Y^{\{\alpha\}} = \emptyset$$

and

$$W \cap X^{\{\alpha\}} = O_m \cap X^{\{\alpha\}} \neq \emptyset.$$

This contradicts lemma 4.1.6.

For (c) notice that $X^{\{\alpha\}}$ is locally compact if and only if $X^{\{\alpha+1\}} = \emptyset$. So (c) follows directly from (a). □

Theorem 4.1.7 is a useful theorem. In the remaining part of this section we give some applications of it. We will prove for ℓ_p-equivalent spaces X and Y that if α is a finite prime component, then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are ℓ_p-equivalent pairs (proposition 4.1.9), and if α is an infinite countable prime component and X and Y are zero-dimensional separable metric spaces, then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are ℓ_p-equivalent pairs (proposition 4.1.12). We will distinguish between the cases of finite and infinite prime components. Although the result for finite prime components is much stronger than the result for countable infinite prime components, the latter case requires most of the work. We first need the following

4.1.8 LEMMA: *Let X be a first countable space and $\alpha < \omega_1$ an ordinal such that $X^{(\alpha)} \neq \emptyset$. Then there is $K \subset X$ such that $K \approx [1, \omega^\alpha]$.*

PROOF: We prove the lemma by transfinite induction on α. For $\alpha = 0$, it is a triviality. Now suppose the lemma is true for every ordinal $\beta < \alpha$, with $\alpha \geq 1$. Fix $x \in X^{(\alpha)}$.
Case 1: α is a successor, say $\alpha = \beta + 1$.

Choose a sequence $(x_n)_{n \in \mathbb{N}}$ in $X^{(\beta)}$ such that $x_n \to x$, and a decreasing open base $\{U_n : n \in \mathbb{N}\}$ at x such that for each $n \in \mathbb{N}$, $x_n \in V_n = U_n \setminus \overline{U}_{n+1}$. Notice that V_n is open, so $V_n^{(\beta)} = V_n \cap X^{(\beta)}$. Hence, $x_n \in V_n^{(\beta)}$. So by the inductive hypothesis, there are $K_n \subset V_n$ such that $K_n \approx [1, \omega^\beta]$. Notice that for every $n \neq m$, $K_n \cap K_m = \emptyset$. Let $K = \bigcup_{n=1}^{\infty} K_n \cup \{x\}$. Then by theorem 2.2.8, $K \approx [1, \omega^\alpha]$.
Case 2: α is a limit ordinal.

Let $(\beta_n)_n$ be an increasing sequence converging to α. Since $x \in X^{(\alpha)}$, there is a decreasing open base $\{U_n : n \in \mathbb{N}\}$ at x such that if $V_n = U_n \setminus \overline{U}_{n+1}$, then $V_n^{(\beta_n)} \neq \emptyset$. By the inductive hypothesis there are $K_n \subset V_n$ such that $K_n \approx [1, \omega^{\beta_n}]$. Then by theorem 2.2.8, $K = \bigcup_{n=1}^{\infty} K_n \cup \{x\}$ is as required. □

In [9], J.W. Baker gives conditions for a space to have an ordinal interval as a subspace.

We first deal with finite prime components, i.e., the numbers 0 and 1.

4.1.9 PROPOSITION: *Let $\alpha \in \{0, 1\}$ and let X and Y be ℓ_p-equivalent metric spaces. Then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are ℓ_p-equivalent pairs.*

PROOF: By proposition 2.2.2 (a), $X^{(\alpha)}$ is closed in X. Let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism. It suffices to prove that $(X, X^{(\alpha)})$ is $(\phi, 0)$-relative to $(Y, Y^{(\alpha)})$. To this end let U and V be open disjoint in X and W open in Y such that

$$(\operatorname{supp} U) \cap W = \emptyset, \text{ and}$$
$$\operatorname{supp} W \subset U \cup V.$$

Suppose that $W \cap Y^{(\alpha)} \neq \emptyset$ and $\overline{V} \cap X^{(\alpha)} = \emptyset$.
Case 1: $\alpha = 0$.

Since $\overline{V} = \emptyset$, we have $\operatorname{supp} W \subset U$. So by proposition 1.4.3,

$$W \subset \operatorname{supp} \operatorname{supp} W \subset \operatorname{supp} U.$$

Since $(\operatorname{supp} U) \cap W = \emptyset$ this gives $W = \emptyset$, hence we arrived at a contradiction.
Case 2: $\alpha = 1$.

Since $\overline{V} \cap X^{(1)} = \emptyset$, $V = \overline{V}$ consists of isolated points, say $V = \{x_s : s \in S\}$. Choose $y \in W \cap Y^{(1)}$ and let $\{W_m : m \in \mathbb{N}\}$ be a decreasing open base at y in W. By corollary 1.2.15 (a) and lemma 1.2.10, there is $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in S$ such that

$$\operatorname{supp} W_m \cap \{x_s : s \notin \{s_1, \ldots, s_m\}\} = \emptyset.$$

Now let $V' = \{x_{s_1}, \ldots, x_{s_m}\}$. Since $\operatorname{supp} W_m \subset U \cup V'$, it follows that

$$W_m \subset \operatorname{supp} \operatorname{supp} W_m \subset \operatorname{supp} (U \cup V') = \operatorname{supp} U \cup \operatorname{supp} V'.$$

Since $W_m \cap \operatorname{supp} U = \emptyset$, we have $W_m \subset \operatorname{supp} V'$. Because V' is finite, we have by lemma 1.4.1 that W_m is finite. This contradicts the fact that $y \in W_m^{(1)}$. □

4.1.10 THEOREM: *Let X and Y be ℓ_p-equivalent metric spaces, let $\alpha \in \{0, 1\}$, and let β be an ordinal. Then*

(a) $X^{(\alpha, \beta)} = \emptyset$ if and only if $Y^{(\alpha, \beta)} = \emptyset$,

(b) $X^{(\alpha, \beta)}$ is compact if and only if $Y^{(\alpha, \beta)}$ is compact, and

(c) $X^{(\alpha, \beta)}$ is locally compact if and only if $Y^{(\alpha, \beta)}$ is locally compact.

PROOF: This follows directly from proposition 4.1.9 and theorem 4.1.7. □

In section 4.2, we prove that the conditions (a) and (b) are sufficient to obtain an isomorphical classification for countable metric spaces with scattered height less than or equal to ω, i.e., if two such spaces satisfy conditions (a) and (b) in theorem 4.1.10 for all ordinals $\alpha \in \{0, 1\}$ and β, then they are ℓ_p-equivalent.

In our search for ℓ_p-equivalent pairs we now consider pairs $(X, X^{(\alpha)})$ for infinite countable prime components α. We start with the following

4.1.11 LEMMA: *Let X be a metric space and A a closed subspace of X. Let O be an open neighborhood of A in X. Then there is a continuous linear function $\phi: C_0(A) \to C_0(X)$ such that for each $f \in C(A)$,*

$$\phi(f) \restriction A = f,\ \phi(f)(X) \subset \operatorname{conv}(f(A) \cup \{0\}),\ \textit{and}\ \phi(f)(X \setminus O) = \{0\}.$$

PROOF: We will construct a continuous linear function $\phi: C_p(A) \to C_p(X)$ with the required properties. Then by proposition 1.2.19, ϕ considered as a function from $C_0(A) \to C_0(X)$ is also continuous, and hence is as required.

Since $A \cup (X \setminus O)$ is a closed subset of X, there is by theorem 2.3.1, a continuous linear function $\psi: C_p(A \cup (X \setminus O)) \to C_p(X)$ such that for each $f \in C(A \cup (X \setminus O))$ we have $\psi(f) \restriction (A \cup (X \setminus O)) = f$ and $\psi(f)(X) \subset \operatorname{conv} f(A \cup (X \setminus O))$.

For each $f \in C(A)$, define $f^*: A \cup (X \setminus O) \to \mathbb{R}$ by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus O \end{cases}$$

Then f^* is a well-defined continuous function. Define $\theta: C_p(A) \to C_p(A \cup (X \setminus O))$ by $\theta(f) = f^*$. Then θ is a well-defined continuous linear function. Finally define $\phi: C_p(A) \to C_p(X)$ by $\phi = \psi \circ \theta$. Then ϕ is a continuous linear function, and we claim that it is as required. Let $f \in C(A)$. Then

$$\phi(f) \restriction A = \psi(\theta(f)) \restriction A = \theta(f) \restriction A = f,$$
$$\phi(f)(X) = \psi(\theta(f))(X) \subset \operatorname{conv}(\theta(f)(A \cup X \setminus O)) = \operatorname{conv}(f(A) \cup \{0\}), \text{ and}$$
$$\phi(f)(X \setminus O) = \psi(\theta(f))(X \setminus O) = \theta(f)(X \setminus O) = \{0\}.$$

This proves the lemma. □

4.1.12 PROPOSITION: *Let $\alpha < \omega_1$ be a prime component and let X and Y be ℓ_p-equivalent zero-dimensional separable metric spaces. Then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are ℓ_p-equivalent pairs.*

PROOF: By proposition 2.2.2 (a), $X^{(\alpha)}$ is closed in X. Let $\phi: C_p(X) \to C_p(Y)$ be a

linear homeomorphism. Then by corollary 1.2.21, ϕ considered as a function from $C_0(X)$ to $C_0(Y)$ is also a linear homeomorphism. It suffices to prove that $(X, X^{(\alpha)})$ is $(\phi, 0)$-relative to $(Y, Y^{(\alpha)})$. To this end let U and V be open in X and W open in Y such that

$$(\operatorname{supp} U) \cap W = \emptyset, \text{ and}$$
$$\operatorname{supp} W \subset U \cup V.$$

Suppose that $W \cap Y^{(\alpha)} \neq \emptyset$ and $\overline{V} \cap X^{(\alpha)} = \emptyset$. By proposition 4.1.9 we must have $\alpha \geq \omega$. Let $y \in W \cap Y^{(\alpha)}$ and let $\{W_m : m \in \mathbb{N}\}$ be a decreasing clopen base at y in W. By lemma 4.1.4 (a), there is a discrete clopen family $\{V_m : m \in \mathbb{N}\}$ such that $V \subset \bigcup_{m \in \mathbb{N}} V_m$ and for each $m \in \mathbb{N}$, there is $\beta < \alpha$ such that $(V_m)^{(\beta)} = \emptyset$. By corollary 1.2.15 (a) and lemma 1.2.10, there is $m \in \mathbb{N}$ such that

$$\operatorname{supp} W_m \cap \bigcup_{i>m} V_m = \emptyset.$$

Let $V' = \bigcup_{i=1}^{m} V_{s_i}$ Notice that V' is clopen and $\operatorname{supp} W_m \subset U \cup V'$. Fix $\beta < \alpha$ such that $(V')^{(\beta)} = \emptyset$. Since $W_m^{(\alpha)} = W_m \cap Y^{(\alpha)} \neq \emptyset$, by lemma 4.1.8 there is a set $K \subset W_m$ such that $K \approx [1, \omega^\alpha]$. Let $L = \overline{\operatorname{supp} K} \cap V'$. Then by corollary 1.2.15 (b), L is compact. Furthermore $L \subset V'$. We also have that L is non-empty. Indeed, if $(\operatorname{supp} K) \cap V' = \emptyset$, then $\operatorname{supp} K \subset U$, and so by proposition 1.4.3,

$$K \subset \operatorname{supp} \operatorname{supp} K \subset \operatorname{supp} U.$$

Since $(\operatorname{supp} U) \cap K = \emptyset$, we then have $K = \emptyset$. Contradiction.

By lemma 4.1.11, there is a continuous linear function $\psi_1 : C_0(K) \to C_0(Y)$ such that for each $f \in C(K)$,

$$\psi_1(f) \mid K = f \text{ and } \psi_1(f)(Y \setminus W_m) = \{0\}.$$

Again by lemma 4.1.11, there is a continuous linear function $\psi_2 : C_0(L) \to C_0(X)$ such that for each $f \in C(L)$,

$$\psi_2(f) \mid L = f \text{ and } \psi_2(f)(X \setminus V') = \{0\}.$$

Define

$$\psi : C_0(K) \to C_0(L) \text{ by } \psi(f) = \phi^{-1}(\psi_1(f)) \mid L, \text{ and}$$
$$\theta : C_0(L) \to C_0(K) \text{ by } \theta(f) = \phi(\psi_2(f)) \mid K.$$

Observe that ψ and θ are linear.

CLAIM: For each $f \in C(K)$, $\theta(\psi(f)) = f$.

Suppose there is $f \in C(K)$ such that $\theta(\psi(f)) \neq f$. Then

$$\phi(\psi_2(\psi(f)))\restriction K \neq f = \psi_1(f)\restriction K.$$

So by corollary 1.4.2 (b),

$$\psi_2(\psi(f))\restriction \operatorname{supp} K \neq \phi^{-1}(\psi_1(f))\restriction \operatorname{supp} K.$$

Since $\psi_1(f)(Y \setminus W_m) = \{0\}$ and $\operatorname{supp} U \subset Y \setminus W_m$, it follows from corollary 1.4.2 (b), that $\phi^{-1}(\psi_1(f))(U) = \{0\}$. Since $U \setminus V' \subset X \setminus V'$, $\psi_2(\psi(f))((\operatorname{supp} K \setminus V') = \{0\}$. Hence

$$\psi(f) = \psi_2(\psi(f))\restriction L \neq \phi^{-1}(\psi_1(f))\restriction L = \psi(f).$$

Contradiction and the claim is proved.

From the claim we conclude that ψ is a linear embedding. Since $L \subset V'$, we have $L^{(\beta)} = \emptyset$. Since L is separable metric, it is countable by the Cantor-Bendixson theorem and so by theorem 2.2.8, there is $\gamma < \beta$ and $n \in \mathbb{N}$ such that $L \approx [1, \omega^\gamma \cdot n]$. Since by theorem 2.4.1 $C_0([1, \omega^\gamma \cdot n]) \sim C_0([1, \omega^\gamma])$, we have a linear embedding $\psi\colon C_0([1, \omega^\alpha]) \to C_0([1, \omega^\gamma])$. By lemma 2.6.7 and the fact that α is a prime component it follows that $\alpha \leq \gamma$. This gives a contradiction since $\gamma < \beta < \alpha$. □

REMARK: a) For $\alpha < \omega_1$ not a prime component, there are ℓ_p-equivalent countable metric spaces X and Y such that $X^{(\alpha)} = \emptyset$, $Y^{(\alpha)} \neq \emptyset$. So $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are not ℓ_p-equivalent pairs. For example let $X = [1, \omega^{\alpha'}]$ and let $Y = [1, \omega^\alpha]$. Then by theorem 2.4.7, X and Y are ℓ_p-equivalent. Since α is not a prime component, $\alpha' < \alpha$ hence $X^{(\alpha)} = \emptyset$. However $Y^{(\alpha)} \neq \emptyset$.

b) The question arises whether "being ℓ_p-equivalent pairs" is independent from the choice of the linear homeomorphism. From the proof of proposition 4.1.12 it follows that for any linear homeomorphism ϕ between $C_p(X)$ and $C_p(Y)$ we have that $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are ℓ_p-equivalent pairs.

4.1.13 THEOREM: *Let X and Y be ℓ_p-equivalent zero-dimensional separable metric spaces and let α, β be ordinals with $\alpha < \omega_1$ a prime component. Then*

(a) $X^{(\alpha, \beta)} = \emptyset$ if and only if $Y^{(\alpha, \beta)} = \emptyset$,

(b) $X^{(\alpha, \beta)}$ is compact if and only if $Y^{(\alpha, \beta)}$ is compact, and

(c) $X^{(\alpha, \beta)}$ is locally compact if and only if $Y^{(\alpha, \beta)}$ is locally compact.

PROOF: This follows directly from proposition 4.1.12 and theorem 4.1.7. □

4.1.14 COROLLARY: *Let X and Y be ℓ_p-equivalent zero-dimensional separable metric spaces and let $\alpha < \omega_1$ be a prime component. Then*

(*a*) $X^{(\alpha)} = \emptyset$ *if and only if* $Y^{(\alpha)} = \emptyset$,
(*b*) $X^{(\alpha)}$ *is compact if and only if* $Y^{(\alpha)}$ *is compact, and*
(*c*) $X^{(\alpha)}$ *is locally compact if and only if* $Y^{(\alpha)}$ *is locally compact.*

PROOF: This is an application of theorem 4.1.13: take $\beta = 0$. □

The strength of theorem 4.1.7 has now become clear. Once we have ℓ_p-equivalent pairs such as in propositions 4.1.9 and 4.1.12, we immediately get ω_1 ℓ_p-equivalent properties.

Although we were not able to prove proposition 4.1.12 for arbitrary metric spaces, we can give for this class of spaces a direct proof of corollary 4.1.14 (a).

4.1.15 THEOREM: *Let X and Y be ℓ_p-equivalent metric spaces and let $\alpha < \omega_1$ be a prime component. Then $X^{(\alpha)} = \emptyset$ if and only if $Y^{(\alpha)} = \emptyset$.*

PROOF: By theorem 4.1.10 we may assume that $\alpha \geq \omega$. Suppose $X^{(\alpha)} = \emptyset$ and $Y^{(\alpha)} \neq \emptyset$. Choose $y \in Y^{(\alpha)}$ and let $\{W_n : n \in \mathbb{N}\}$ be an open decreasing base at y.

CLAIM 1: There is a locally finite open covering $\{V_s : s \in S\}$ of X such that for each $s \in S$, there is $\beta < \alpha$ such that $\overline{V}_s \cap X^{(\beta)} = \emptyset$.

Since $X^{(\alpha)} = \emptyset$, $\mathcal{U} = \{X \setminus X^{(\beta)} : \beta < \alpha\}$ is an open covering of X. Let $\{V_s : s \in S\}$ be a locally finite open covering of X such that $\{\overline{V}_s : s \in S\}$ refines $\mathcal{U}$. Then for $s \in S$ there is $\beta < \alpha$ such that $\overline{V}_s \subset X \setminus X^{(\beta)}$. Hence $\overline{V}_s \cap X^{(\beta)} = \emptyset$.

Fix β as in the claim. Let $\{F_s : s \in S\}$ be a closed covering of X such that for each $s \in S$, $F_s \subset V_s$. By corollary 1.2.15 (a) and lemma 1.2.10, there are $m \in \mathbb{N}$ and $\{s_1, \ldots, s_m\} \subset S$ such that

$$\operatorname{supp} W_m \cap \bigcup_{s \notin \{s_1, \ldots, s_m\}} V_s = \emptyset.$$

Let $V' = \bigcup_{i=1}^{n} V_{s_i}$ and $F' = \bigcup_{i=1}^{n} F_{s_i}$. Then V' is open, F' is closed and $F' \subset V'$. Find a copy K of $[1, \omega^\alpha]$ in W_m (lemma 4.1.8). Let $L = \overline{\operatorname{supp} K}$. Note that $\operatorname{supp} K \subset F'$, hence $L \subset V'$. Furthermore L is compact. If $L = \emptyset$, then $\operatorname{supp} K = \emptyset$, hence $K = \emptyset$. This gives a contradiction, so $L \neq \emptyset$. Let $\psi_1 : C_0(K) \to C_0(Y)$, $\psi_2 : C_0(L) \to C_0(X)$, $\psi : C_0(K) \to C_0(L)$ and $\theta : C_0(L) \to C_0(K)$ be continuous linear functions such as in the proof of proposition 4.1.12.

CLAIM 2: For each $f \in C(K)$, $\theta(\psi(f)) = f$.

Suppose there is $f \in C(K)$ such that $\theta(\psi(f)) \neq f$. Then

$$\phi(\psi_2(\psi(f))) \mid K \neq f = \psi_1(f) \mid K,$$

so by corollary 1.4.2 (b),

$$\psi_2(\psi(f)) \mid \operatorname{supp} K \neq \phi^{-1}(\psi_1(f)) \mid \operatorname{supp} K.$$

This implies that

$$\psi(f) = \psi_2(\psi(f)) \mid L \neq \phi^{-1}(\psi_1(f)) \mid L = \psi(f).$$

Contradiction, and the claim is proved.

From the claim we conclude that ψ is a linear embedding. Since $L \subset \overline{V'}$, we have by proposition 2.2.4,

$$L^{(\beta)} \subset L \cap \overline{V'}^{(\beta)} \subset L \cap \overline{V'} \cap X^{(\beta)} = \emptyset.$$

As in proposition 4.1.12, we arrive at a contradiction. □

By the Cantor-Bendixson theorem, each scattered separable metric space has scattered height less than ω_1. Hence we have the following

4.1.16 COROLLARY: *Let X and Y be l_p-equivalent separable metric spaces. Then X is scattered if and only if Y is scattered.*

The question arises whether corollary 4.1.16 holds in the class of metric spaces, or whether theorem 4.1.15 holds for all prime components. In the proof of theorem 4.1.15 we used the isomorphical classification of function spaces of countable compact spaces (cf. section 2.4). For prime components larger than or equal to ω_1 we cannot use this result. Moreover, we are not able to use larger compact ordinal intervals (cf. section 2.6) because they are not metric. There is one case in which we can overcome these technical problems.

4.1.17 THEOREM: *Let X and Y be l_p-equivalent metric spaces. Then $X^{(\omega_1)} = \emptyset$ if and only if $Y^{(\omega_1)} = \emptyset$.*

PROOF: The proof is almost the same as the proof of theorem 4.1.15. Replace α in the proof of theorem 4.1.15 by ω_1. Copy this proof until the set K is introduced. We have $\overline{V'} \cap X^{(\beta)} = \emptyset$ for some $\beta < \omega_1$. Find a prime component $\alpha < \omega_1$, such that $\beta < \alpha$. Find a copy K of $[1, \omega^\alpha]$ in W_m and let $L = \overline{\operatorname{supp} K}$. As in theorem 4.1.15 we obtain a

contradiction. □

We finish this section by posing the following

Question: Let X and Y be ℓ_p-equivalent metric spaces. For which ordinals α is theorem 4.1.15 true? Is it true for prime components? Is it only true for prime components? Is it true for $\omega_1 \cdot 2$?

§4.2. An isomorphical classification

In this section we present an isomorphical classification of function spaces of countable metric spaces which have scattered height less than or equal to ω. In chapter 2 we have considered finite spaces. *We will assume in this section that all spaces are infinite.*

Let X be a space. For ordinals α and β, we define the following:

$X(\alpha, \beta) = 0$ if and only if $X^{(\alpha, \beta)} = \emptyset$,
$X(\alpha, \beta) = 1$ if and only if $X^{(\alpha, \beta)}$ is non-empty and compact, and
$X(\alpha, \beta) = 2$ if and only if $X^{(\alpha, \beta)}$ is not compact.

With this notation, part of the results in section 4.1 can be reformulated as follows:

(I) Let X and Y be ℓ_p-equivalent spaces zero-dimensional separable metric spaces. Then for every pair of ordinals α, β with $\alpha < \omega_1$ a prime component, we have $X(\alpha, \beta) = Y(\alpha, \beta)$.

As mentioned above we restrict ourselves in this section to countable metric spaces X which have scattered height less than or equal to ω. In this class of spaces, (I) takes the following form (note that for such X, we have $X^{(\omega)} = \emptyset$, so the ordinals we have to consider here are the finite ordinals):

(II) Let X and Y be ℓ_p-equivalent countable metric spaces which have scattered height less than or equal to ω. Then for every $n \in \mathbb{N} \cup \{0\}$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$.

In this section we will show that the necessary conditions in (II) are also sufficient, i.e., if for two infinite countable metric spaces X and Y which have scattered height less than

or equal to ω, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$ for every $n \in \mathbb{N} \cup \{0\}$, then X and Y are ℓ_p-equivalent.

Before we consider function spaces of countable metric spaces, we first deal with the countable metric spaces itself. Some of the next lemmas are formulated in a more general form in case their proofs do not use special properties of countable metric spaces.

4.2.1 LEMMA: *Let X be a space. Then for every $n \in \mathbb{N}$,*
$X^{(0, n)} \subset X^{(1, n-1)} \subset X^{(0, n-1)}$.

PROOF: It is easily seen that

$$X^{(0, 1)} \subset X^{(1)} \subset X = X^{(0)},$$

from which it follows that for every $n \in \mathbb{N}$,

$$(X^{(0, 1)})^{(0, n-1)} \subset (X^{(1)})^{(0, n-1)} \subset X^{(0, n-1)}.$$

This completes the proof of this lemma. □

4.2.2 COROLLARY: *Let X be a space, such that there is $n \in \mathbb{N}$ with $X(0, n) = 0$. Let $n_0 = \min\{n : X(0, n) = 0\}$. Then $n_1 = \min\{n : X(1, n) = 0\}$ is well-defined and $n_0 = n_1$ or $n_0 = n_1 + 1$.* □

PROOF: By lemma 4.2.1, $X^{(1, n_0)} \subset X^{(0, n_0)}$, so that $n_1 \leq n_0$. Again by lemma 4.2.1, $X^{(0, n_1+1)} \subset X^{(1, n_1)}$, so that $n_0 \leq n_1 + 1$. □

We can distinguish the spaces X with scattered height less than or equal to ω into two types. The first type consists of those X such that for each $n \in \mathbb{N}$, $X(0, n) = 2$; then also $X(1, n) = 2$ for each $n \in \mathbb{N}$, by lemma 4.2.1. For the other spaces, there is $n \in \mathbb{N}$ such that $X(0, n) = 1$ or $X(0, n) = 0$; if $X(0, n) = 1$, then $X(0, n + 1) = 0$, so in both cases $X(0, n) = 0$ for some $n \in \mathbb{N}$. For spaces of the second type we have

4.2.3 LEMMA: *Let X be a space such that there is $m \in \mathbb{N}$ with $X(0, m) = 0$. Then there is $n \in \mathbb{N}$ such that X satisfies one of the following conditions:*

$(a)_n$ $X(0, n) = 0$, $X(0, n-1) = 1$, *and* $X(1, n-1) = 1$,
$(b)_n$ $X(0, n) = 0$, $X(0, n-1) = 2$, *and* $X(1, n-1) = 1$,
$(c)_n$ $X(0, n) = 0$, $X(0, n-1) = 2$, *and* $X(1, n-1) = 2$,
$(d)_n$ $X(1, n) = 0$, $X(1, n-1) = 1$, *and* $X(0, n) = 1$,

$(e)_n$ $X(1, n)=0, X(1, n-1)=2$, *and* $X(0, n)=1$,
$(f)_n$ $X(1, n)=0, X(1, n-1)=2$, *and* $X(0, n)=2$, *or*
(g) $X(0, 1)=0, X(0, 0)=2$, *and* $X(1, 0)=0$,
(*i.e., X is an infinite discrete space*).

PROOF: As in corollary 4.2.2, let

$$n_0=\min\{n: X(0, n)=0\}, \text{ and } n_1=\min\{n: X(1, n)=0\}.$$

Then $n_0=n_1$ or $n_0=n_1+1$. Since $X\neq\emptyset$, $n_0\in\mathbb{N}$.
case 1: $n_0=n_1$.

In this case $X(0, n_0)=X(1, n_0)=0$, $X(0, n_0-1)\neq 0$ and $X(1, n_0-1)\neq 0$. If $X(0, n_0-1)=1$, then by lemma 4.2.1, $X(1, n_0-1)=1$, so X satisfies $(a)_{n_0}$. If $X(0, n_0-1)=2$, then X satisfies $(b)_{n_0}$ or $(c)_{n_0}$.
case 2: $n_0=n_1+1$, where $n_1\in\mathbb{N}$.

In this case $X(0, n_1+1)=X(1, n_1)=0$, $X(0, n_1)\neq 0$ and $X(1, n_1-1)\neq 0$. If $X(1, n_1-1)=1$, then by lemma 4.2.1, $X(0, n_1)=1$, so X satisfies $(d)_{n_1}$. If $X(1, n_1-1)=2$, then X satisfies $(e)_{n_1}$ or $(f)_{n_1}$.
case 3: $n_1=0$, and $n_0=1$.

Since $X(1, 0)=0$, X has no accumulation points, so in this case, X is an infinite discrete space. □

At the end of this section we will show that for each case, there exist countable metric spaces with scattered height less than or equal to ω satisfying the corresponding conditions. We will now present a special decomposition of the spaces of interest in this section. (cf. corollaries 4.2.5, 4.2.7 and 4.2.9). We restrict ourselves to countable metric spaces.

4.2.4 LEMMA: *Let X be a countable metric space. Let A and B be closed in X with* $A\subset B$ *and suppose that* $A(0, 1)=B(0, 1)=1$. *Then there is a decreasing clopen base* $\{U_n: n\in\mathbb{N}\}$ *at* $B^{(0, 1)}$ *in X such that* $U_1=X$ *and* $(U_n\setminus U_{n+1})\cap A$ *is not compact for every* $n\in\mathbb{N}$.

PROOF: Since $B^{(0, 1)}$ is compact, there is a decreasing clopen base $\{V_n: n\in\mathbb{N}\}$ at $B^{(0, 1)}$ in X. We now inductively find the U_n. Let $U_1=X$ and suppose we have chosen $U_1, \ldots, U_n$ for some $n\in\mathbb{N}$. Since $A^{(0, 1)}\subset B^{(0, 1)}$, U_n is a neighborhood of $A^{(0, 1)}$. But then $U_n\cap A$ is not compact, from which it follows that there is an infinite closed discrete set E in $U_n\cap A$. Since $B^{(0, 1)}$ is compact, without loss of generality we may as-

sume that $E \cap B^{(0,1)} = \emptyset$, so there is $i > n$ such that $V_i \subset X \setminus E$. If we now let $U_{n+1} = V_i$, then $E \subset (U_n \setminus U_{n+1}) \cap A$. □

4.2.5 COROLLARY: *Let X be a countable metric space, and let $m \in \mathbb{N}$.*

(a) If $X(0, m) = X(1, m) = 1$, then there is a clopen decreasing base $\{U_n : n \in \mathbb{N}\}$ at $X^{(0,m)}$ in X, such that $U_1 = X$ and $(U_n \setminus U_{n+1})(1, m-1) = 2$ for every $n \in \mathbb{N}$.

(b) If $X(1, m) = X(0, m+1) = 1$, then there is a clopen decreasing base $\{U_n : n \in \mathbb{N}\}$ at $X^{(1,m)}$ in X, such that $U_1 = X$ and $(U_n \setminus U_{n+1})(0, m) = 2$ for every $n \in \mathbb{N}$.

PROOF: This is a direct consequence of lemmas 4.2.1 and 4.2.4. □

4.2.6 LEMMA: *Let X be a countable metric space. Let A and B be closed in X with $A \subset B$. If A and B are locally compact but not compact, then X can be written as a clopen disjoint union $X = \bigcup_{i=1}^{\infty} X_i$ such that for each i, $X_i \cap A$ and $X_i \cap B$ are compact and non-empty.*

PROOF: Since B is locally compact but not compact and X is zero-dimensional, we can write X as a clopen disjoint union $X = \bigcup_{i=1}^{\infty} K_i$ such that for every $i \in \mathbb{N}$, $K_i \cap B$ is compact. Since $A \subset B$, for every $i \in \mathbb{N}$, $A \cap K_i$ is compact as well. Since A is not compact we can find a strictly increasing sequence $(i_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, $A \cap K_{i_n}$ is not empty. Taking $X_n = \bigcup_{i=i_{n-1}}^{i_n} K_i$ (where $i_0 = 1$) we obtain the desired decomposition. □

4.2.7 COROLLARY: *Let X be a countable metric space and let $m \in \mathbb{N}$.*

(a) If $X(0, m) = 0$ and $X(1, m-1) = 2$, then X can be written as a clopen disjoint union $X = \bigcup_{i=1}^{\infty} A_i$ such that for every $i \in \mathbb{N}$, $A_i(0, m-1) = A_i(1, m-1) = 1$.

(b) If $X(1, m) = 0$ and $X(0, m) = 2$, then X can be written as a clopen disjoint union $X = \bigcup_{i=1}^{\infty} A_i$ such that for every $i \in \mathbb{N}$, $A_i(0, m) = A_i(1, m-1) = 1$.

PROOF: This a direct consequence of lemmas 4.2.1 and 4.2.6. □

4.2.8 LEMMA: *Let X be a countable metric space. Let A and B be closed in X with $A \subset B$. If A is compact and non-empty and B is locally compact but not compact, then X can be written as a clopen disjoint union $X = X_1 \cup X_2$ such that*

(1) $X_1 \cap B$ *is compact and non-empty and*
(2) $X_2 \cap A = \emptyset$.

PROOF: As in the proof of lemma 4.2.6, X can be written as a clopen disjoint union $X = \bigcup_{i=1}^{\infty} K_i$ with for each $i \in \mathbb{N}$, $K_i \cap B$ is compact and non-empty. Since A is compact, there is i_0 such that $A \cap \bigcup_{i>i_0} K_i = \emptyset$. Now let $X_1 = \bigcup_{i=1}^{i_0} K_i$ and $X_2 = \bigcup_{i>i_0} K_i$. □

4.2.9 COROLLARY: *Let X be a countable metric space and let $m \in \mathbb{N} \cup \{0\}$.*
(*a*) *If $X(0, m) = 2$, $X(0, m+1) = 0$ and $X(1, m) = 1$, then X can be written as a clopen disjoint union $X = X_1 \cup X_2$ such that $X_1(0, m) = 1$ and $X_2(1, m) = 0$.*
(*b*) *If $X(1, m) = 2$, $X(1, m+1) = 0$ and $X(0, m+1) = 1$, then X can be written as a clopen disjoint union $X = X_1 \cup X_2$ such that $X_1(1, m) = 1$ and $X_2(0, m+1) = 0$.*

PROOF: This is a direct consequence of lemmas 4.2.1 and 4.2.8. □

REMARK: Notice that in corollary 4.2.9 (a) we also have that $X_1(1, m) = 1$ because $X^{(1, m)} = X_1^{(1, m)} \cup X_2^{(1, m)}$. Similarly we have $X_2(0, m+1) = 0$, $X_2(0, m) = 2$ and if $m \neq 0$, $X_2(1, m-1) = 2$. In addition, in corollary 4.2.9 (b) we have $X_1(0, m+1) = 1$, $X_2(1, m+1) = 0$, $X_2(1, m) = 2$ and $X_2(0, m) = 2$.

We return to the subject of function spaces. The following lemma together with the "decomposition" lemmas above will play a fundamental role in proving the announced isomorphical classification of function spaces of infinite countable metric spaces which have scattered height less than or equal to ω.

4.2.10 LEMMA: *Let X be a countable metric space, A a non-empty compact subspace of X, and $\{U_n : n \in \mathbb{N}\}$ a clopen decreasing base at A in X such that $U_1 = X$. Let Y be a countable metric space, B a non-empty compact subspace of Y, and $\{V_n : n \in \mathbb{N}\}$ a clopen decreasing base at B in Y such that $V_1 = X$. Let $k \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$, $\phi_n : C_p(U_n \setminus U_{n+1}) \to C_p(V_n \setminus V_{n+1})$ is a linear k-mapping. Define $\phi : C_{p,A}(X) \to C_{p,B}(Y)$ by*

$$\phi(f) \mid V_n \setminus V_{n+1} = \phi_n(f \mid U_n \setminus U_{n+1}) \text{ and } \phi(f) \mid B = 0.$$

Then ϕ is a well-defined linear k-mapping. If moreover each ϕ_n is a linear k-homeomorphism, then ϕ is a linear k-homeomorphism.

PROOF: To prove that ϕ is well-defined it suffices to prove that $\phi(f)$ is continuous at points of B. Let $\varepsilon > 0$. Since $f(A) = 0$, there is an open neighborhood W of A with $f(W) \subset (-\frac{\varepsilon}{k}, \frac{\varepsilon}{k})$. There is $n_0 \in \mathbb{N}$ such that $A \subset U_{n_0} \subset W$, so

$$f(U_{n_0}) \subset (-\frac{\varepsilon}{k}, \frac{\varepsilon}{k}).$$

Then it easily follows by k-linearity of ϕ_n for every n, that $\phi(f)(V_{n_0}) \subset (-\varepsilon, \varepsilon)$, so that $\phi(f)$ is continuous at points of B.

To prove continuity of ϕ, it suffices to prove that ϕ is continuous at 0. Let $P \subset Y$ be finite and $\varepsilon > 0$. For each $n \in \mathbb{N}$, let $P_n = P \cap (V_n \setminus V_{n+1})$. Since P is finite, there is $n_0 \in \mathbb{N}$ such that for each $n > n_0$, $P_n = \emptyset$. Let $n \leq n_0$. Since ϕ_n is continuous, there are a finite $Q_n \subset U_n \setminus U_{n+1}$ and $\delta_n > 0$ such that

$$\phi_n(<0, Q_n, \delta_n>) \subset <0, P_n, \varepsilon>.$$

Let $Q = \bigcup_{n=1}^{n_0} Q_n$ and $\delta = \min\{\delta_i : 1 \leq i \leq n_0\}$. Then it is easily seen that

$$\phi(<0, Q, \delta>) \subset <0, P, \varepsilon>.$$

The k-linearity of ϕ is an easy exercise.

Now suppose each ϕ_n is a linear k-homeomorphism. Define $\psi: C_{p,B}(Y) \to C_{p,A}(X)$ by

$$\psi(f) \mid U_n \setminus U_{n+1} = \phi_n^{-1}(f \mid V_n \setminus V_{n+1}) \text{ and } \psi(f) \mid A = 0.$$

Then ψ is a well-defined linear k-mapping which is easily seen to be equal to ϕ^{-1}, hence ϕ is a linear k-homeomorphism. □

We are now in a position to prove an isomorphical classification of function spaces of countable metric spaces which have scattered height less than or equal to ω. First we consider the case of countable metric spaces which have scattered height strictly less than ω. The proof will be an inductive one. In the following two lemmas we deal with spaces at a "low level". The space T in lemma 4.2.12 is the one defined in section 3.3.

4.2.11 LEMMA: *Let $q \in \mathbb{N}$. There is $k_q \in \mathbb{N}$ such that if X and Y are infinite countable compact spaces with $\kappa(X), \kappa(Y) \leq q$, then $C_p(X) \overset{k_q}{\sim} C_p(Y)$.*

PROOF: Let X be an infinite countable compact metric space with $\kappa(X) \leq q$. By theorem 2.2.8, there are $1 \leq m \leq q$ and $n \in \mathbb{N}$ such that $X \approx [1, \omega^m \cdot n]$. Let $a = \omega^m$ and

$A = X^{(m)}$. Notice that $A = \{\omega^m \cdot i : 1 \leq i \leq n\}$. Then

$$C_p(X) \overset{2}{\sim} C_{p,\,a}([1, \omega^m]) \times C_p(A) \qquad \text{(corollaries 2.2.10 and 2.3.4)}$$
$$\overset{1}{\sim} C_{p,\,a}(A \oplus [1, \omega^m]) \qquad \text{(lemma 2.3.6)}$$
$$\overset{1}{\sim} C_{p,\,a}([1, \omega^m]) \qquad \text{(lemma 2.3.5)}$$
$$\overset{2}{\sim} C_p([1, \omega^m]) \qquad \text{(lemma 2.3.9).}$$

So that $C_p(X) \overset{4}{\sim} C_p([1, \omega^m])$.
To finish the lemma it suffices to prove the following

CLAIM: There is $l \in \mathbb{N}$ such that for every $1 \leq r \leq q$ we have

$$C_p([1, \omega^r]) \overset{l}{\sim} C_p([1, \omega]).$$

Let $1 \leq r \leq q$. By theorem 2.4.7, there is a linear homeomorphism $\phi: C_p([1, \omega^r]) \to C_p([1, \omega])$. Then by corollary 1.2.21, $\phi: C_0([1, \omega^r]) \to C_0([1, \omega])$ is also a linear homeomorphism. Since these two function spaces are Banach spaces, there is $l(r) \in \mathbb{N}$ such that for every $f \in C_0([1, \omega^r])$ we have

$$\frac{1}{l(r)}\|f\| < \|\phi(f)\| < l(r)\|f\|.$$

Then $l = \max\{l(r) : r \leq q\}$ is as required. □

From now on we fix for each $q \in \mathbb{N}$, k_q as in lemma 4.2.11.

4.2.12 LEMMA: *Let $q \in \mathbb{N}$. There is $l_q \geq k_q$ such that if X and Y are countable metric spaces with $\kappa(X), \kappa(Y) \leq q$, $X(0, 1) = Y(0, 1) = 1$ and $X(1, 0) = Y(1, 0) = 1$, then*

$$C_p(X) \overset{l_q}{\sim} C_p(Y).$$

PROOF: Let X be a countable metric space with $\kappa(X) \leq q$ and $X(0, 1) = X(1, 0) = 1$. Let $A = X^{(1)}$. Then by assumption A is compact. Since X/A is a perfect image of X, and X is not locally compact, we have that X / A is a non-locally compact countable metric space with exactly one non-isolated point. It then easily follows that X/A is homeomorphic to the space T. Then by corollary 2.3.4,

$$C_p(X) \overset{2}{\sim} C_{p,\,\infty}(T) \times C_p(A).$$

If A is finite, then $T \oplus A$ is homeomorphic to T, so $C_p(X) \overset{2}{\sim} C_{p,\,\infty}(T)$. Now suppose that A is infinite. We have by lemma 4.2.11, $C_p(A) \overset{k_q}{\sim} C_p([1, \omega])$. Note that by the above argument $C_p(T) \overset{2}{\sim} C_{p,\,\infty}(T)$, so that $C_p(X) \overset{4 \cdot k_q}{\sim} C_p(T \oplus [1, \omega])$. Since $(T \oplus [1, \omega])^{(1)}$ is

finite, the same argument gives $C_p(T \oplus [1, \omega]) \overset{2}{\sim} C_{p,\infty}(T)$. We conclude that $C_p(X) \overset{8 \cdot k_q}{\sim} C_p(T)$. Then $l_q = (8 \cdot k_q)^2$ is as required. □

From now on we fix for each $q \in \mathbb{N}$, l_q as in lemma 4.2.12.

The isomorphical classification for countable metric spaces with scattered height less than ω will be given after the following lemma, in which all the tools developed in this section are used.

4.2.13 LEMMA: *For every $q \in \mathbb{N}$, there is $r_q \in \mathbb{N}$, such that if X and Y are infinite countable metric spaces with $\kappa(X) \leq q$, $\kappa(Y) \leq q$ and for every $n \in \mathbb{N}$, $X(0, n) = Y(0, n)$, and $X(1, n) = Y(1, n)$, then $C_p(X) \overset{r_q}{\sim} C_p(Y)$.*

PROOF: Let $q \in \mathbb{N}$ and for every $m \leq q$, let $s_m = 4^m \cdot l_q$. Let X and Y be infinite countable metric spaces with $\kappa(X) \leq q$, $\kappa(Y) \leq q$ and for every $n \in \mathbb{N}$, $X(0, n) = Y(0, n)$, and $X(1, n) = Y(1, n)$. Since $X^{(q)} = Y^{(q)} = \emptyset$, $X(0, q) = Y(0, q) = 0$, hence X and Y both satisfy the condition in lemma 4.2.3. If X and Y do not satisfy (g), find $m \in \mathbb{N}$ such that X and Y both satisfy one of the cases $(a)_m$ through $(f)_m$. Notice that $1 \leq m \leq q$. We will prove that $C_p(X) \overset{s_m}{\sim} C_p(Y)$ by induction on m. Then $r_q = s_q$ is as required. The inductive proof is organized as follows: We prove case $(a)_m$ for $m = 1$ and for $m > 1$ we use $(c)_{m-1}$. The proof of case $(b)_m$ makes use of case $(a)_m$ and for $m > 1$ it makes use of case $(f)_{m-1}$. We prove case $(c)_m$ using case $(a)_m$. We prove case $(d)_m$ for $m = 1$ and for $m > 1$ we use $(f)_{m-1}$ For case $(e)_m$ we use $(d)_m$ and if $m > 1$ we use $(c)_m$. Finally in case $(f)_m$ we use $(d)_m$.

case $(a)_m$: $X(0, m) = Y(0, m) = 0$, $X(0, m-1) = Y(0, m-1) = 1$, and
$X(1, m-1) = Y(1, m-1) = 1$.

Notice that in this case we have $X(1, m) = Y(1, m) = 0$. For $m = 1$ we have that X and Y are infinite countable compact spaces, so by lemma 4.2.11, $C_p(X) \overset{k_q}{\sim} C_p(Y)$. Note that $k_q \leq l_q \leq s_1$. For $m > 1$, let $A = X^{(0, m-1)}$ and $B = Y^{(0, m-1)}$. By proposition 2.3.2,

$$C_p(X) \overset{2}{\sim} C_{p,A}(X) \times C_p(A) \text{ and } C_p(Y) \overset{2}{\sim} C_{p,B}(Y) \times C_p(B). \tag{1}$$

Define $Z_1 = X \oplus A$ and $Z_2 = Y \oplus B$. Notice that since $m > 1$,

$$Z_1^{(0, m-1)} = X^{(0, m-1)}, Z_1^{(1, m-1)} = X^{(1, m-1)},$$
$$Z_2^{(0, m-1)} = Y^{(0, m-1)}, \text{ and } Z_2^{(1, m-1)} = Y^{(1, m-1)}.$$

Let $C = Z_1^{(0,\,m-1)}$ and $D = Z_2^{(0,\,m-1)}$. Then by (1) and by lemma 2.3.6,

$$C_p(X) \overset{2}{\sim} C_{p,\,C}(Z_1) \text{ and } C_p(Y) \overset{2}{\sim} C_{p,\,D}(Z_2).$$

By corollary 4.2.5 (a), there are clopen decreasing bases $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ at C and D, respectively such that $U_1 = Z_1$, $V_1 = Z_2$,

$$(U_n \setminus U_{n+1})(1, m-2) = 2 \text{ and } (V_n \setminus V_{n+1})(1, m-2) = 2.$$

Notice that then also

$$(U_n \setminus U_{n+1})(0, m-2) = (V_n \setminus V_{n+1})(0, m-2) = 2.$$

It is easily seen that

$$(U_n \setminus U_{n+1})(0, m-1) = 0 \text{ and } (V_n \setminus V_{n+1})(0, m-1) = 0.$$

Then $(c)_{m-1}$ gives $C_p(U_n \setminus U_{n+1}) \overset{s_{m-1}}{\sim} C_p(V_n \setminus V_{n+1})$ for every $n \in \mathbb{N}$, whence by lemma 4.2.10, $C_{p,\,C}(Z_1) \overset{s_{m-1}}{\sim} C_{p,\,D}(Z_2)$. In conclusion we have $C_p(X) \overset{s_m}{\sim} C_p(Y)$. This completes the proof of case $(a)_m$.

case $(b)_m$: $X(0, m) = Y(0, m) = 0$, $X(0, m-1) = Y(0, m-1) = 2$, and
$X(1, m-1) = Y(1, m-1) = 1$.

Again in this case we have $X(1, m) = Y(1, m) = 0$. By corollary 4.2.9 (a), X and Y can be written as clopen disjoint unions, $X = A \cup B$ and $Y = C \cup D$ such that $A(0, m-1) = C(0, m-1) = 1$ and $B(1, m-1) = D(1, m-1) = 0$. By the remark following corollary 4.2.9 we now have by case $(a)_m$, $C_p(A) \overset{s_m}{\sim} C_p(C)$ and for $m > 1$, by $(f)_{m-1}$, $C_p(B) \overset{s_{m-1}}{\sim} C_p(D)$. If $m = 1$ then B and D are infinite discrete and so $C_p(B) \overset{1}{\sim} C_p(D)$. With lemma 2.3.7 it now follows that $C_p(X) \overset{s_m}{\sim} C_p(Y)$. This completes the proof of case $(b)_m$.

case $(c)_m$: $X(0, m) = Y(0, m) = 0$, $X(0, m-1) = Y(0, m-1) = 2$, and
$X(1, m-1) = Y(1, m-1) = 2$.

Again $X(1, m) = Y(1, m) = 0$. We have by corollary 4.2.7 (a), X and Y can be written as clopen disjoint unions, $X = \bigcup_{i=1}^{\infty} A_i$ and $Y = \bigcup_{i=1}^{\infty} B_i$ such that for each $i \in \mathbb{N}$, $A_i(0, m-1) = B_i(0, m-1) = A_i(1, m-1) = B_i(1, m-1) = 1$. By case $(a)_m$, we then have $C_p(A_i) \overset{s_m}{\sim} C_p(B_i)$, so that by lemma 2.3.7, $C_p(X) \overset{s_m}{\sim} C_p(Y)$. This completes the proof of case $(c)_m$.

case $(d)_m$: $X(1, m) = Y(1, m) = 0$, $X(1, m-1) = Y(1, m-1) = 1$, and
$X(0, m) = Y(0, m) = 1$.

The proof of this case is almost the same as the proof of case $(a)_m$. Instead of $(c)_{m-1}$ we use $(f)_{m-1}$, instead of lemma 4.2.11, we use lemma 4.2.12 and instead of corollary 4.2.5 (a), we use corollary 4.2.5 (b).

case $(e)_m$: $X(1, m) = Y(1, m) = 0$, $X(1, m-1) = Y(1, m-1) = 2$, and
$X(0, m) = Y(0, m) = 1$.

The proof is almost the same as the proof of case $(b)_m$. Instead of corollary 4.2.9 (a), we use corollary 4.2.9 (b), instead of $(a)_m$ we use $(d)_m$ and instead of $(f)_{m-1}$ we use $(c)_m$.

case $(f)_m$: $X(1, m) = Y(1, m) = 0$, $X(1, m-1) = Y(1, m-1) = 2$, and
$X(0, m) = Y(0, m) = 2$.

The proof is almost the same as the proof of case $(c)_m$. Instead of corollary 4.2.7 (a), we use corollary 4.2.7 (b), and instead of $(a)_m$ we use $(d)_m$.

case (g): X and Y are infinite discrete spaces.

Since X and Y are countable we have that X and Y are both homeomomorphic to $\mathbb{N}$. Now apply lemma 2.3.5.

This completes the proof of this lemma. □

4.2.14 THEOREM: *Let X and Y be infinite countable metric spaces with $\kappa(X), \kappa(Y) < \omega$ such that for every $n \in \mathbb{N}$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$. Then $C_p(X) \sim C_p(Y)$.*

PROOF: Let $q = \max(\kappa(X), \kappa(Y))$ and apply lemma 4.2.13. □

We have completed the case of countable metric spaces with scattered height less than ω, so from now on we consider spaces with scattered height equal to ω. So let X be a countable metric space with $\kappa(X) = \omega$. As mentioned in the beginning of this section, there are two cases:

(a) there is $n \in \mathbb{N}$, such that $X(0, n) = 0$,
(b) for every $n \in \mathbb{N}$, $X(0, n) = 2$.

Note that if a space X satisfies condition (b), then $\kappa(X) \geq \omega$.

We will first deal with spaces for which case (a) holds. We need another decomposition lemma.

4.2.15 LEMMA: *Let X and Y be countable metric spaces such that $\kappa(X)=\omega$, $\kappa(Y)\leq\omega$, $X(0,n)=Y(0,n)$ and $X(1,n)=Y(1,n)$ for every $n\in\mathbb{N}\cup\{0\}$ and such that case (a) holds for X. Then X and Y can be written as clopen disjoint unions, $X=\bigcup_{i=1}^{\infty}X_i$ and $Y=\bigcup_{i=1}^{\infty}Y_i$ such that $\kappa(X_i),\kappa(Y_i)<\omega$ and for every $i,n\in\mathbb{N}$, $X_i(0,n)=Y_i(0,n)$ and $X_i(1,n)=Y_i(1,n)$.*

PROOF: Since X satisfies (a), and $\kappa(X)=\omega$, there is by lemma 4.2.3, $k\in\mathbb{N}$ such that X and Y both satisfy one of the cases $(a)_k$ through $(f)_k$. We prove the lemma by induction on k and the inductive proof is organized as the inductive proof in lemma 4.2.13.

case $(a)_m$: $X(0,m)=Y(0,m)=0$, $X(0,m-1)=Y(0,m-1)=1$, and
$X(1,m-1)=Y(1,m-1)=1$.

Since $\kappa(X)=\omega$, X is not compact. This implies $m>1$. By corollary 4.2.5 (a), there are clopen decreasing bases $\{U_n:n\in\mathbb{N}\}$ and $\{V_n:n\in\mathbb{N}\}$ at $X^{(0,m-1)}$ and $Y^{(0,m-1)}$ respectively, such that $U_1=X$ and $V_1=Y$,

$$(U_n\setminus U_{n+1})(1,m-2)=2 \text{ and } (V_n\setminus V_{n+1})(1,m-2)=2.$$

CLAIM: There is $l\in\mathbb{N}$ such that $\kappa(U_l)<\omega$.

Since $\kappa(X)=\omega$, $\mathcal{U}=\{X\setminus X^{(n)}:n\in\mathbb{N}\}$ is an open cover of X without finite subcover. Since X is zero-dimensional, there is a disjoint clopen refinement $\{A_i:i\in\mathbb{N}\}$ of $\mathcal{U}$. Since $X^{(0,m-1)}$ is compact, there is n such that $X^{(0,m-1)}\subset\bigcup_{i=1}^{n}A_i$. There is $l\in\mathbb{N}$ such that $U_l\subset\bigcup_{i=1}^{n}A_i$, and this l satisfies the claim.

Without loss of generality we may assume that also $\kappa(V_l)<\omega$. Now let $X_1=U_l$ and $Y_1=V_l$. Notice that

$$X_1(0,n)=Y_1(0,n) \text{ and } X_1(1,n)=Y_1(1,n) \text{ for every } n\in\mathbb{N},$$
$$(X\setminus U_l)(1,m-2)=(Y\setminus V_l)(1,m-2)=2 \text{ and}$$
$$(X\setminus U_l)(0,m-1)=(Y\setminus V_l)(0,m-1)=0.$$

So by $(c)_{m-1}$ we have that X and Y can be written as clopen disjoint unions, $X\setminus U_l=\bigcup_{i=2}^{\infty}X_i$ and $Y\setminus V_k=\bigcup_{i=2}^{\infty}Y_i$ such that for every $i\geq 2$ and $n\in\mathbb{N}$, $X_i(0,n)=Y_i(0,n)$ and $X_i(1,n)=Y_i(1,n)$ and the lemma has been proved in this case.

case $(b)_m$: $X(0,m)=Y(0,m)=0$, $X(0,m-1)=Y(0,m-1)=2$, and

$$X(1, m-1) = Y(1, m-1) = 1.$$

By corollary 4.2.9 (a), X and Y can be written as clopen disjoint unions $X = A \cup B$ and $Y = C \cup D$ such that $A(0, m-1) = C(0, m-1) = 1$ and $B(1, m-1) = D(1, m-1) = 0$. By the remark following corollary 4.2.9 we now have in cases of scattered height ω by case $(a)_m$ or by case $(f)_{m-1}$, the desired decomposition of X and Y.

case $(c)_m$: $X(0, m) = Y(0, m) = 0$, $X(0, m-1) = Y(0, m-1) = 2$, and
$X(1, m-1) = Y(1, m-1) = 2$.

We have by corollary 4.2.7 (a), X and Y are clopen disjoint unions $X = \bigcup_{i=1}^{\infty} A_i$ and $Y = \bigcup_{i=1}^{\infty} B_i$ such that $A_i(0, m-1) = B_i(0, m-1) = A_i(1, m-1) = B_i(1, m-1) = 1$. By case $(a)_m$ (applied in cases where A_i or B_i has scattered height ω), we have the desired decomposition of X and Y.

case $(d)_m$: $X(1, m) = Y(1, m) = 0$, $X(1, m-1) = Y(1, m-1) = 1$, and
$X(0, m) = Y(0, m) = 1$.

The proof of this case is almost the same as the proof of case $(a)_m$. Instead of corollary 4.2.5 (a), we use corollary 4.2.5 (b), and instead of $(c)_{m-1}$ we use $(f)_{m-1}$.

case $(e)_m$: $X(1, m) = Y(1, m) = 0$, $X(1, m-1) = Y(1, m-1) = 2$, and
$X(0, m) = Y(0, m) = 1$.

The proof is almost the same as the proof of case $(b)_m$. Instead of corollary 4.2.9 (a), we use corollary 4.2.9 (b), instead of $(a)_m$ we use $(d)_m$ and instead of $(f)_{m-1}$ we use $(c)_m$.

case $(f)_m$: $X(1, m) = Y(1, m) = 0$, $X(1, m-1) = Y(1, m-1) = 2$, and
$X(0, m) = Y(0, m) = 2$.

The proof is almost the same as the proof of case $(c)_m$. Instead of corollary 4.2.7 (a), we use corollary 4.2.7 (b) and instead of $(a)_m$ we use $(d)_m$.

This completes the proof of this lemma. □

4.2.16 THEOREM: *Let X and Y be countable metric spaces such that $\kappa(X) = \omega$, $\kappa(Y) \leq \omega$ and for every $n \in \mathbb{N} \cup \{0\}$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$. If X is a space satisfying (a), then X and Y are l_p-equivalent.*

PROOF: This follows directly from theorem 4.2.14, lemmas 2.3.7 and 4.2.15. □

This theorem completes the case for spaces satisfying (a). So now we only have to consider spaces satisfying (b). Again a decomposition of these spaces is needed. This will be given in a claim in the following:

4.2.17 THEOREM: *Let X and Y be countable metric spaces such that $\kappa(X)=\kappa(Y)=\omega$ and both satisfying (b). Then X and Y are l_p-equivalent.*

PROOF: We begin with the following:

CLAIM: We can write $X=\bigcup_{i=1}^{\infty}X_i$ and $Y=\bigcup_{i=1}^{\infty}Y_i$ as clopen disjoint unions, such that there are sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ such that $n_i+1<m_i$, $m_i+1<n_{i+1}$, $X_i(1,n_i)\neq 0$, $X_i(1,n_{i+1})=0$, $Y_i(1,m_i)\neq 0$ and $Y_i(1,m_{i+1})=0$.

It is easily seen that $\{X\setminus X^{(1,n)}:n\in\mathbb{N}\}$ is an open cover of X without finite subcover. Since X is countable, there is a clopen disjoint refinement $\{A_i:i\in\mathbb{N}\}$ of this cover. We may assume that there exists a strictly increasing sequence $(k_i)_{i\in\mathbb{N}}$ of natural numbers such that for each $i\in\mathbb{N}$, $A_i(1,k_i)\neq 0$ and $A_i(1,k_i+1)=0$ (note that X satisfies condition (b), so take unions of the A_i's). In the same way Y can be written as a clopen disjoint union $Y=\bigcup_{i=1}^{\infty}B_i$ such that there are $l_1<l_2\cdots$ with $B_i(1,l_i)\neq 0$ and $B_i(1,l_i+1)=0$ for each $i\in\mathbb{N}$. Now let $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be subsequences of $(k_i)_{i\in\mathbb{N}}$ and $(l_i)_{i\in\mathbb{N}}$, respectively, such that $n_i+1<m_i$, $m_i+1<n_{i+1}$. By letting X_i be a appropriate finite union of the A_j's and the same for the Y_i's, we are done.

Let $Z=X_1\oplus Y_1\oplus X_2\oplus Y_2\oplus\cdots$.
Because $n_i+1<m_i$, $(X_i\oplus Y_i)(0,n)=Y_i(0,n)$ and $(X_i\oplus Y_i)(1,n)=Y_i(1,n)$ for every $n\in\mathbb{N}\cup\{0\}$. Both $X_i\oplus Y_i$ and Y_i satisfy (a), so by theorem 4.2.14 or theorem 4.2.16, $C_p(X_i\oplus Y_i)\sim C_p(Y_i)$, so that $C_p(Z)\sim C_p(Y)$. By interchanging the role of X and Y we also have $C_p(Z)\sim C_p(X)$. We conclude that $C_p(X)\sim C_p(Y)$. □

Since we have considered all possible cases we can now formally state the result announced at the beginning of this section.

4.2.18 THEOREM: *Let X and Y be infinite countable metric spaces, such that $\kappa(X),\kappa(Y)\leq\omega$. Then X and Y are l_p-equivalent if and only if for every $n\in\mathbb{N}\cup\{0\}$, $X(0,n)=Y(0,n)$ and $X(1,n)=Y(1,n)$.*

PROOF: This follows immediately from theorems 4.2.14, 4.2.16, 4.2.17 and 4.1.13. □

The question naturally arises whether theorem 4.2.18 can be generalized to all countable metric spaces. One is tempted to conjecture the following:

> Let X and Y be countable metric spaces. Then X and Y are ℓ_p-equivalent if and only if for every prime component α and ordinal β we have $X(\alpha, \beta) = Y(\alpha, \beta)$.

In section 4.3 we will show that this conjecture is false.

In this section we saw that an infinite countable metric space X, with scattered height less than or equal to ω, satisfies one of the conditions in lemma 4.2.3 or for every $n \in \mathbb{N}$, $X(0, n) = X(1, n) = 2$. The question remains whether each of the considered classes is non-empty. We will prove that in each case there are ω-many spaces satisfying the given conditions (except for case (g) in lemma 4.2.3 of course).

For convenience, for every $n \in \mathbb{N}$ define $S_n = [1, \omega^n]$. Let X be a space. We define $T(X)$ to be the space obtained from T by replacing each isolated point of T by a copy of X. Each copy of X will then be clopen in $T(X)$. Inductively we define $T^k(X) = T(T^{k-1}(X))$ for $k > 1$, and let $T^0(X) = X$. Similarly $S_k(X)$ will be the space obtained from S_k by replacing each isolated point of S_k by a copy of X.

4.2.19 LEMMA: *Let X be a non-discrete space. Then for every $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$, $S_n(X)(0, m) = X(0, m)$ and $S_n(X)(1, m) = X(1, m)$.*

PROOF: Since X is non-empty we have for every $n \in \mathbb{N}$, $(S_n(X))^{(1)} = S_n(X^{(1)})$. Since X is non-discrete, we have

$$S_n(X)(0, 0) = X(0, 0) \text{ and } S_n(X)(1, 0) = X(1, 0).$$

Since S_n is compact, we also have for every $m \in \mathbb{N}$,

$$S_n(X)(0, m) = X(0, m) \text{ and } S_n(X)(1, m) = X(1, m). \square$$

If X is a scattered space we have for $n \neq m$ that $S_n(X)$ and $S_m(X)$ are not homeomorphic. So this lemma implies that we only have to give one countable metric space for each of the cases mentioned above, since the lemma then immediately gives ω-many. For every $n \in \mathbb{N}$ define $X_n = T^{n-1}(T)$, and $Y_n = T^{n-1}(S_1)$. Let X_0 be any one-point space.

4.2.20 PROPOSITION: *For every $n \in \mathbb{N}$, we have*

(a) Y_n satisfies the conditions in lemma 4.2.3 $(a)_n$,

(*b*) $Y_n \oplus (X_{n-1} \times \mathbb{N})$ *satisfies the conditions in lemma 4.2.3* $(b)_n$,
(*c*) $Y_n \times \mathbb{N}$ *satisfies the conditions in lemma 4.2.3* $(c)_n$,
(*d*) X_n *satisfies the conditions in lemma 4.2.3* $(d)_n$,
(*e*) $X_n \oplus (Y_n \times \mathbb{N})$ *satisfies the conditions in lemma 4.2.3* $(e)_n$,
(*f*) $X_n \times \mathbb{N}$ *satisfies the conditions in lemma 4.2.3* $(f)_n$, *and*
(*g*) $Y = \oplus_{n=1}^{\infty} Y_n$ *satisfies* $Y(0, m) = Y(1, m) = 2$ *for each* $m \in \mathbb{N}$.

PROOF: We start this proof with the following

CLAIM: Let X be a non-discrete space, and let $m \in \mathbb{N} \cup \{0\}$. If $X(0, m) = 1$, then $T(X)(0, m+1) = 1$, and if $X(1, m) = 1$, then $T(X)(1, m+1) = 1$.

Notice that since X is non-empty, $T(X)^{(0,m)} = T(X^{(0,m)})$. Furthermore because $X^{(0,m)}$ is compact and non-empty, $T(X)^{(0,m+1)}$ contains only one point, hence $T(X)(0, m+1) = 1$. Since X is a non-discrete space, $T(X)^{(1)} = T(X^{(1)})$, so the second part follows from the fist part. This proves the claim.

Since $Y_1 = S_1$ satisfies the conditions in lemma 4.2.3 $(a)_1$, we have by the claim that Y_n satisfies the conditions in lemma 4.2.3 $(a)_n$. We can prove (*d*) similarly. Case (c) follows easily from (a) and case (f) easily follows from (d). It is easily seen that $Y_1 \oplus (X_0 \times \mathbb{N})$ satisfies the conditions in lemma 4.2.3 $(b)_1$. For $n > 1$, case (b) follows from (a) and (f). Case (e) is a combination of (d) and (c). Finally (g) follows from (d) since for each $n > 2$, $Y_n(0, n-2) = Y_n(1, n-2) = 2$. □

Of course the spaces constructed above are not the only possible ones. In fact one can replace isolated points in T by other countable metric spaces to obtain more examples.

§4.3. More ℓ_p-equivalent properties of metric spaces

The ℓ_p-equivalent pairs found in section 4.1 do not provide a complete isomorphical classification for the function spaces $C_p(X)$, with X countable and metric. In this section we present two different types of ℓ_p-equivalent properties which show that an isomorphical classification for these function spaces must be more complicated than the one derived in section 4.2, where we dealt with countable metric spaces of scattered height less than or equal to ω.

Let $\alpha \geq \omega$ be a countable prime component and let β be a countable ordinal. Recall from section 4.1 that for any space X,

$$X^{\ll \alpha, \beta \gg} = \bigcap_{\gamma < \alpha} X^{(\gamma, \beta)}.$$

Let $\alpha \geq \omega$ be a countable prime component and let γ, β be countable ordinals. By $X^{\ll \alpha, \beta, \gamma \gg}$ we denote the set $X^{\{\gamma\}}$ with respect to the pair $(X, X^{\ll \alpha, \beta \gg})$. Notice that if γ is a successor, say $\gamma = \delta + 1$, then we have $X^{\ll \alpha, \beta, \gamma \gg} = (X^{\ll \alpha, \beta, \delta \gg})^{(0, 1)}$.

The numbers $X \ll \alpha, \beta, \gamma \gg$ are defined for ordinals α, β and γ similarly to the numbers $X(\alpha, \beta)$ as follows:

$X \ll \alpha, \beta, \gamma \gg = 0$ if and only if $X^{\ll \alpha, \beta, \gamma \gg} = \emptyset$,
$X \ll \alpha, \beta, \gamma \gg = 1$ if and only if $X^{\ll \alpha, \beta, \gamma \gg}$ is non-empty and compact, and
$X \ll \alpha, \beta, \gamma \gg = 2$ if and only if $X^{\ll \alpha, \beta, \gamma \gg}$ is not compact.

For a subset A of a space X we will denote by $C_{0,A}(X)$ the subspace $\{f \in C(X) : f(A) = \{0\}\}$ of $C_0(X)$.

4.3.1 PROPOSITION: *Let X and Y be zero-dimensional separable metric l_p-equivalent spaces and $\alpha \geq \omega$ a countable prime component. Then $(X, X^{\ll \alpha, 1 \gg})$ and $(Y, Y^{\ll \alpha, 1 \gg})$ are l_p-equivalent pairs.*

PROOF: Let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism. It suffices to prove that $(X, X^{\ll \alpha, 1 \gg})$ is $(\phi, 0)$-relative to $(Y, Y^{\ll \alpha, 1 \gg})$. Let U and V be open subsets of X and W an open subset of Y, such that

$(\operatorname{supp} U) \cap W = \emptyset$, and
$\operatorname{supp} W \subset U \cup V$,

Suppose $W \cap Y^{\ll \alpha, 1 \gg} \neq \emptyset$ and $\overline{V} \cap X^{\ll \alpha, 1 \gg} = \emptyset$.

Let $(\gamma_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of ordinals such that $\gamma_n \to \alpha$ $(n \to \infty)$. Let $y \in W \cap Y^{\ll \alpha, 1 \gg}$ and let $\{W_n \mid n \in \mathbb{N}\}$ be a clopen decreasing base at y in W. For each $n \in \mathbb{N}$, $y \in W_n^{(\gamma_n, 1)}$, hence we may assume that for each $n \in \mathbb{N}$ we can find a closed copy of $[1, \omega^{\gamma_n}] \times \mathbb{N}$ in $W_n \setminus W_{n+1}$. By lemma 4.1.4, there is a discrete clopen family $\{A_i : i \in \mathbb{N}\}$ such that $V \subset \bigcup_{i=1}^{\infty} A_i$ and such that for every $i \in \mathbb{N}$, there is $\gamma_i < \alpha$ with $A_i^{(\gamma_i, 1)} = \emptyset$. By corollary 1.2.15 (a) and lemma 1.2.10, there is $p \in \mathbb{N}$ such that

$$\operatorname{supp} W_p \cap \bigcup_{i > p} A_i = \emptyset.$$

Let $A = \bigcup_{i=1}^{p} A_i$. Then there is $\gamma < \alpha$ such that $A^{(\gamma, 1)} = \emptyset$. By corollary 4.1.5, there is a

discrete clopen family $\{B_i : i \in \mathbb{N}\}$ such that $A = \bigcup_{i=1}^{\infty} B_i$ and such that for every $i \in \mathbb{N}$, $B_i^{(\gamma)}$ is compact. By corollary 1.2.15 (a) and lemma 1.2.10 there is $k > p$ such that

$$\operatorname{supp} W_k \cap \bigcup_{i>k} B_i = \emptyset.$$

Let $B = \bigcup_{i=1}^{k} B_i$. Then $B^{(\gamma)}$ is compact. We now have

$$(\operatorname{supp} U) \cap W_k = \emptyset, \text{ and}$$
$$\operatorname{supp} W_k \subset U \cup B,$$

Since $W_k^{(\alpha)} \neq \emptyset$, we have by proposition 4.1.12, that $B^{(\alpha)} \neq \emptyset$, so that $B^{(\gamma)} \neq \emptyset$. This implies that $\overline{\operatorname{supp} B^{(\gamma)}}$ is a non-empty compactum in Y.

Since $\overline{\operatorname{supp} B^{(\gamma)}}$ is compact, this implies that there exists a closed copy L_n of $[1, \omega^{\gamma_n}]$ in W_n such that $\overline{\operatorname{supp} B^{(\gamma)}} \cap L_n = \emptyset$. If $\operatorname{supp} L_n \cap B = \emptyset$, then $\operatorname{supp} L_n \subset U$ hence by proposition 1.4.3, $L_n \subset \operatorname{supp} \operatorname{supp} L_n \subset \operatorname{supp} U$. This implies $L_n = \emptyset$, contradiction. Hence $\operatorname{supp} L_n \cap B \neq \emptyset$. Let

$$M = (\overline{\operatorname{supp} B^{(\gamma)}} \cap W_k) \cup \{y\},$$
$$L_\alpha = \bigcup_{n>k} L_n \cup \{y\},$$
$$L = L_\alpha \cup M, \text{ and}$$
$$K = (\overline{\operatorname{supp} L} \cap B) \cup B^{(\gamma)}.$$

By lemma 4.1.11, there is a continuous linear function $\eta_1 : C_0(L) \to C_0(Y)$ such that for each $f \in C(L)$,

$$\eta_1(f) \restriction L = f \text{ and } \eta_1(f)(Y \setminus W_k) = \{0\}.$$

Again by lemma 4.1.11, there is a continuous linear function $\eta_2 : C_0(K) \to C_0(X)$ such that for each $f \in C(K)$,

$$\eta_2(f) \restriction K = f \text{ and } \eta_2(f)(X \setminus B) = \{0\}.$$

Define

$$\theta : C_{0,M}(L) \to C_{0,B^{(m)}}(K) \text{ by } \theta(f) = \phi^{-1}(\eta_1(f)) \restriction K, \text{ and}$$
$$\psi : C_0(K) \to C_0(L) \text{ by } \psi(f) = \phi(\eta_2(f)) \restriction L.$$

Let $f \in C_{0,M}(L)$. Then $\eta_1(f)(Y \setminus W_k) = \{0\}$. Since $\overline{\operatorname{supp} B^{(\gamma)}} \cap W_k \subset M$, we have that $\eta_1(f)(\operatorname{supp} B^{(\gamma)}) = \{0\}$. So by corollary 1.4.2, $\phi^{-1}(\eta_1(f))(B^{(\gamma)}) = \{0\}$, hence $\theta(f)(B^{(\gamma)}) = \{0\}$. Now it is easily seen that both θ and ψ are well-defined continuous linear functions.

CLAIM: θ is a linear embedding.

It suffices to prove that for $f \in C_{0,M}(L)$, we have $\psi(\theta(f)) = f$. Suppose to the contrary that for some $f \in C_{0,M}(L)$, we have $\psi(\theta(f)) \neq f$. This implies that

$$\phi(\eta_2(\theta(f))) \restriction L \neq \eta_1(f) \restriction L.$$

Then by corollary 1.4.2,

$$\eta_2(\theta(f)) \restriction \operatorname{supp} L \neq \phi^{-1}(\eta_1(f)) \restriction \operatorname{supp} L.$$

Now $\eta_2(\theta(f)) \equiv 0$ on $X \setminus B$, and $\eta_1(f) \equiv 0$ on $Y \setminus W_k$. Since $\operatorname{supp} U \subset Y \setminus W_k$, we have by corollary 1.4.2 that $\phi^{-1}(\eta_1(f)) \equiv 0$ on U. Since $\operatorname{supp} L \subset U \cup B$, we have

$$\eta_2(\theta(f)) \restriction K \neq \phi^{-1}(\eta_1(f)) \restriction K.$$

Hence $\theta(f) \neq \theta(f)$. Contradiction. This proves the claim.

Since $L_\alpha \cap M = \{y\}$ it follows that $C_{0,M}(L)$ is linearly homeomorphic to $C_{0,\{y\}}(L_\alpha)$. Note that $L_\alpha \approx [1, \omega^\alpha]$. By lemma 2.3.9 and remark 2.3.10, $C_{0,\{y\}}(L_\alpha)$ is linearly homeomorphic to $C_0(L_\alpha)$. By lemma 2.2.9, the space $B/B^{(\gamma)}$ obtained from B by identifying $B^{(\gamma)}$ to a single point a has scattered height $\gamma+1$. Let $Z = K/B^{(\gamma)}$. Then $\kappa(Z) \leq \gamma+1$, so there are $\beta \leq \gamma$ and $m \in \mathbb{N}$ such that $Z \approx [1, \omega^\beta \cdot m]$. By lemma 2.3.3 and remark 2.3.10, $C_{0,\{a\}}(Z)$ is linearly homeomorphic to $C_{0,B^{(\gamma)}}(K)$. Furthermore by lemma 2.3.9 and remark 2.3.10, $C_{0,\{a\}}(Z)$ is linearly homeomorphic to $C_0(Z)$. Now the claim implies that we have a linear embedding from $C_0(L_\alpha)$ to $C_0(Z)$, or equivalently a linear embedding from $C_0[1, \omega^\alpha]$ to $C_0[1, \omega^\gamma]$. Since $\gamma < \alpha$ we have a contradiction by lemma 2.6.7. □

4.3.2 COROLLARY: *Let X and Y be ℓ_p-equivalent zero-dimensional separable metric spaces, let $\alpha \geq \omega$ be a countable prime component and let γ be a countable ordinal. Then*

(a) $X^{\ll \alpha, 1, \gamma \gg} = \emptyset$ if and only if $Y^{\ll \alpha, 1, \gamma \gg} = \emptyset$,

(b) $X^{\ll \alpha, 1, \gamma \gg}$ is compact if and only if $Y^{\ll \alpha, 1, \gamma \gg}$ is compact, and

(c) $X^{\ll \alpha, 1, \gamma \gg}$ is locally compact if and only if $Y^{\ll \alpha, 1, \gamma \gg}$ is locally compact.

PROOF: This follows directly from proposition 4.3.1 and theorem 4.1.7. □

We will now give an example of two countable metric metric spaces X and Y such that for every pair of ordinals α, β with α a countable prime component, $X(\alpha, \beta) = Y(\alpha, \beta)$, and such that X and Y are not ℓ_p-equivalent.

Let X be the space obtained from T by replacing each $(i, j) \in T$ by $[1, \omega^j]$ $(i, j \in \mathbb{N})$ and let $Y = T([1, \omega]) \oplus [1, \omega^\omega]$ (for definitions see section 4.2). Let p be any point.

Then

$$X^{(1)} \approx X \text{ and } Y^{(1)} \approx T \oplus [1, \omega^\omega],$$
$$X^{(0,1)} \approx \{p\} \text{ and } Y^{(0,1)} \approx \{p\},$$
$$X^{(1,1)} \approx \{p\} \text{ and } Y^{(1,1)} \approx \{p\}, \text{ and}$$
$$X^{(\omega)} \approx \{p\} \text{ and } Y^{(\omega)} \approx \{p\}.$$

So for every pair of ordinals α, β with α a countable prime component, we have $X(\alpha, \beta) = Y(\alpha, \beta)$. However by corollary 4.3.2, X and Y are not ℓ_p-equivalent since

$$X^{\ll\omega, 1\gg} \approx \{p\} \text{ and } Y^{\ll\omega, 1\gg} = \emptyset.$$

Note that the scattered height of X and Y is $\omega+1$, so a classification such as in section 4.2 does not hold when we consider countable metric spaces with scattered height one higher than ω. In this example only $\alpha=0$, $\alpha=1$, $\alpha=\omega$, $\beta=0$ and $\beta=1$ are necessary. It is possible to build more complex examples in which higher ordinals are involved.

Question 1: Let $\alpha \geq \omega$ be a countable prime component, and let X and Y be ℓ_p-equivalent zero-dimensional separable metric spaces. For which ordinals β do we have that $(X, X^{\ll\alpha, \beta\gg})$ and $(Y, Y^{\ll\alpha, \beta\gg})$ are ℓ_p-equivalent pairs?

By propositions 4.1.12 and 4.3.1 we have a positive answer to this question for $\beta=0$ and $\beta=1$. We conjecture that this question has a positive answer for all ordinals β.

By corollary 4.3.2 and question 1 one could think that for two countable metric spaces X and Y which satisfy $X(\alpha, \beta) = Y(\alpha, \beta)$ and $X\ll\alpha, \beta, \gamma\gg = Y\ll\alpha, \beta, \gamma\gg$, for all ordinals α, β and γ with α a prime component, we have that X and Y are ℓ_p-equivalent. In the sequel we will give an example which shows that this is not the case. We first need a new notion which will be used in proposition 4.3.7. We present it in a general setting since it seems to be interesting in itself.

Let X and Y be spaces and let $\phi: C_p(X) \to C_p(Y)$ be a linear function. Let A be a non-empty closed subset of X and U a neighborhood of A in X. Let B be a non-empty closed subset of Y and V a neighborhood of B in Y. Finally let $m \in \mathbb{N}$. We say that the triple (U, V, m) is ***relatively bounded*** with respect to the triple (A, B, ϕ) whenever for each $g \in C(X)$ satisfying

(a) $g((X \setminus U) \cup A) = \{0\}$,
(b) $g(U) \subset (-1/m, 1/m)$, and
(c) $\phi(g)(B) = \{0\}$,

we have

(d) $\phi(g)(V) \subset (-1, 1)$.

4.3.3 LEMMA: *Let X and Y be spaces and let $\phi: C_p(X) \to C_p(Y)$ be a continuous linear function. Let A be a non-empty closed subset of X and U a neighborhood of A in X. Let B be a non-empty closed subset of Y and V a neighborhood of B in Y. Let $m \in \mathbb{N}$. Suppose (U, V, m) is relatively bounded with respect to (A, B, ϕ). If U_1 is a neighborhood of A such that $U_1 \subset U$, V_1 is a neighborhood of B such that $V_1 \subset V$ and $k \geq m$, then (U_1, V_1, k) is also relatively bounded with respect to (A, B, ϕ).*

PROOF: Let $g \in C(X)$ be such that $g((X \setminus U_1) \cup A) = \{0\}$, $g(U_1) \subset (-1/k, 1/k)$, and $\phi(g)(B) = \{0\}$. Then obviously $g((X \setminus U) \cup A) = \{0\}$. For $z \in U \setminus U_1$, $g(z) = 0$, hence $g(U) \subset (-1/m, 1/m)$. So $\phi(g)(V) \subset (-1, 1)$, hence $\phi(g)(V_1) \subset (-1, 1)$. □

4.3.4 LEMMA: *Let X and Y be metric spaces and let $\phi: C_p(X) \to C_p(Y)$ be a continuous linear function. Let A be a non-empty compact subset of X and let B be a non-empty compact subset of Y. Then there are a neighborhood U of A in X, a neighborhood V of B in Y and $m \in \mathbb{N}$ such that (U, V, m) is relatively bounded with respect to (A, B, ϕ).*

PROOF: Suppose the lemma is false. Let $\{U_n\}_{n \in \mathbb{N}}$ be an open base at A in X such that for each $n \in \mathbb{N}$, $\overline{U}_{n+1} \subset U_n$. Let $\{V_n\}_{n \in \mathbb{N}}$ be an open decreasing base at B in Y. By induction we construct $\{k_i : i \in \mathbb{N}\} \subset \mathbb{N}$, $\{g_i : i \in \mathbb{N}\} \subset C(X)$ and $\{y_i : i \in \mathbb{N}\} \subset Y$ such that

(1) $1 = k_1 < k_2 < k_3 < \cdots$,

and for every $i \in \mathbb{N}$,

(2) $g_i((X \setminus U_{k_i}) \cup A) = \{0\}$,

(3) for every $j \leq i$, $g_j(U_{k_i}) \subset (-\frac{1}{i^2}, \frac{1}{i^2})$,

(4) $\phi(g_i)(B) = \{0\}$,

(5) $y_i \in V_{k_i}$ and $|\phi(g_i)(y_i)| \geq 1$,

(6) for every $j < i$, $\phi(g_j)(V_{k_i}) \subset (-\frac{1}{2(i-1)}, \frac{1}{2(i-1)})$, and

(7) for every $j < i$, $\phi(g_i)(y_j) = 0$.

Let $k_1 = 1$. By assumption there is $g_1 \in C(X)$ such that $g_1((X \setminus U_{k_1}) \cup A) = \{0\}$, $g_1(U_{k_1}) \subset (-1, 1)$, $\phi(g_1)(B) = \{0\}$ and $\phi(g_1)(V_{k_1}) \not\subset (-1, 1)$. Let $y_1 \in V_{k_1}$ be such that

$|\phi(g_1)(y_1)| \geq 1$.

Let $m \geq 1$ and suppose we found $k_1, \ldots, k_m, g_1, \ldots, g_m$ and $y_1, \ldots, y_m$. For each $j \leq m$ we have $g_j(A) = 0$ and $\phi(g_1)(B) = \{0\}$. By continuity of $g_1, \ldots, g_m$, there is $k_{m+1} > k_m$ such that

(8) for each $j \leq m$, $g_j(U_{k_{m+1}}) \subset (-\frac{1}{(m+1)^2}, \frac{1}{(m+1)^2})$,

(9) $U_{k_{m+1}} \cap \text{supp}\{y_1, \ldots, y_m\} \subset A$, and

(10) for each $j \leq m$, $\phi(g_j)(V_{k_{m+1}}) \subset (-\frac{1}{2m}, \frac{1}{2m})$.

Again by assumption there is $g_{m+1} \in C(X)$ such that $g_{m+1}((X \setminus U_{k_{m+1}}) \cup A) = \{0\}$,

$$g_{m+1}(U_{k_{m+1}}) \subset (-\frac{1}{(m+1)^2}, \frac{1}{(m+1)^2}),$$

$\phi(g_{m+1})(B) = \{0\}$ and $\phi(g_{m+1})(V_{k_{m+1}}) \not\subset (-1, 1)$. Let $y_{m+1} \in V_{k_{m+1}}$ be such that $|\phi(g_{m+1})(y_{m+1})| \geq 1$. To complete the inductive construction we have to verify (7) for $i = m+1$. Since $g_{m+1}(\text{supp}(y_j)) = \{0\}$ for $j \leq m$, we have by corollary 1.4.2, $\phi(g_{m+1})(y_j) = 0$. This completes the inductive construction.

Now let $g = \Sigma_{i=1}^{\infty} g_i$. We will show that $g \in C(X)$. For $i \in \mathbb{N}$ and $z \notin \overline{U}_{k_{i+1}}$, we have by (2), $g(z) = \Sigma_{j=1}^{i} g_j(z)$. So $g \mid X \setminus A$ is well-defined and continuous. It remains to prove that g is continuous at points of A. Since $g(A) = 0$ this follows from the fact that for every $i \in \mathbb{N}$, $g(U_{k_i}) \subset (-1/i, 1/i)$. Indeed let $z \in U_{k_i} \setminus A$. Then there is $j \geq i$ such that $z \in U_{k_j} \setminus U_{k_{j+1}}$. Then by (2), $g(z) = \Sigma_{k=1}^{j} g_j(z)$, and hence by (3),

$$|g(z)| \leq \sum_{k=1}^{j} |g_k(z)| < j \cdot \frac{1}{j^2} \leq \frac{1}{i}.$$

We conclude that $g \in C(X)$. So $\phi(g) = \Sigma_{i=1}^{\infty} \phi(g_i) \in C(Y)$. Since B is compact we may assume that $y_n \to b$ $(n \to \infty)$ for some $b \in B$. By (4), $\phi(g)(b) = 0$, hence $\phi(g)(y_n) \to 0$ $(n \to \infty)$. However for every $i \in \mathbb{N}$, we have

$$\begin{aligned}
|\phi(g)(y_i)| &= |\Sigma_{j=1}^{\infty} \phi(g_j)(y_i)| \\
&= |\Sigma_{j=1}^{i-1} \phi(g_j)(y_i) + \phi(g_i)(y_i) + \Sigma_{j=i+1}^{\infty} \phi(g_j)(y_i)| \\
&= |\Sigma_{j=1}^{i-1} \phi(g_j)(y_i) + \phi(g_i)(y_i)| && \text{by (7)} \\
&\geq |\phi(g_i)(y_i)| - \Sigma_{j=1}^{i-1} |\phi(g_j)(y_j)| \\
&\geq 1 - (i-1) \cdot \frac{1}{2(i-1)} = \frac{1}{2} && \text{by (5) and (6)}
\end{aligned}$$

This contradiction proves the lemma. □

4.3.5 COROLLARY: *Let X and Y be metric spaces and let $\phi: C_p(X) \to C_p(Y)$ be a linear homeomorphism. Let A be a non-empty compact subset of X and let B be a non-empty compact subset of Y. Then there are a neighborhood U of A in X, a neighborhood V of B in Y and $m \in \mathbb{N}$ such that (U, V, m) is relatively bounded with respect to (A, B, ϕ) and (V, U, m) is relatively bounded with respect to (B, A, ϕ^{-1}).*

PROOF: By lemma 4.3.4, there are neighborhoods U_1 and U_2 of A in X, neighborhoods V_1 and V_2 of B in Y, and $m_1, m_2 \in \mathbb{N}$ such that (U_1, V_1, m) is relatively bounded with respect to (A, B, ϕ) and (V_1, U_1, m) is relatively bounded with respect to (B, A, ϕ^{-1}). Let $m = \max(m_1, m_2)$, $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Then by lemma 4.3.3, this U, V and m satisfy the conditions in the lemma. □

Let X and Y be spaces. Let E and F be linear subspaces of $C_0(X)$ resp. $C_0(Y)$. Let $m \in \mathbb{N}$. A linear function $\phi: E \to F$ is said to be a ***linear m-embedding*** whenever ϕ is an embedding and

(1) if $f \in E$ satisfies $f(X) \subset (-1/m, 1/m)$, then $\phi(f)(Y) \subset (-1, 1)$, and
(2) if $f \in E$ satisfies $\phi(f)(Y) \subset (-1/m, 1/m)$, then $f(X) \subset (-1, 1)$.

This definition is comparable with the notion of a linear k-mapping introduced in section 2.3. Since we need linear m-embeddings only in a very specific situation our definition is not in the most general form as was the case in section 2.3.

4.3.6 LEMMA: *Let X and Y be compact spaces. Let $x \in X$ and let $\{U_n : n \in \mathbb{N}\}$ be a clopen decreasing base at $x \in X$ such that $U_1 = X$. Let $y \in Y$ and let $\{V_n : n \in \mathbb{N}\}$ be a clopen decreasing base at $y \in Y$ such that $V_1 = Y$. Let $k \in \mathbb{N}$. Suppose that for every $n \in \mathbb{N}$,*

$$\phi_n: C_0(U_n \setminus U_{n+1}) \to C_0(V_n \setminus V_{n+1})$$

is a linear k-embedding. Define $\phi: C_{0,\{x\}}(X) \to C_{0,\{y\}}(Y)$ by

$$\phi(f) \mid V_n \setminus V_{n+1} = \phi_n(f \mid U_n \setminus U_{n+1}) \text{ and } \phi(f)(y) = 0.$$

Then ϕ is a well-defined linear k-embedding.

PROOF: As in lemma 4.2.10, ϕ is well-defined and linear. To prove continuity of ϕ it suffices to prove that ϕ is continuous at 0. To this end let $P \subset Y$ be compact, let $\varepsilon > 0$ and observe that by linearity of ϕ,

$$\phi(<0, X, \varepsilon/k>) \subset <0, P, \varepsilon>.$$

To prove that ϕ is an embedding, let $P \subset X$ be compact, let $\varepsilon > 0$ and observe that

$$<0, Y, \varepsilon/k> \cap \phi(C_{0,\{x\}}(X)) \subset \phi(<0, P, \varepsilon>). \square$$

We will now give an example of two countable metric metric spaces X and Y such that for all ordinals α, β and γ with α a countable prime component, we have $X(\alpha, \beta) = Y(\alpha, \beta)$, $X \ll \alpha, \beta, \gamma \gg = Y \ll \alpha, \beta, \gamma \gg$, and X and Y are not l_p-equivalent.

Let X be the space obtained from T by replacing each $(i, j) \in T$ by $[1, \omega^i]$ $(i, j \in \mathbb{N})$. Let $Y = S([1, \omega^\omega))$. Let x be the unique point in $X^{(\omega)}$ and y the unique point in $Y^{(\omega)}$. Then

$X^{(1)} \approx X$ and $Y^{(1)} \approx Y$
$X^{(0, 1)} = \{x\}$ and $Y^{(0, 1)} = \{y\}$,
$X^{(1, 1)} = \{x\}$ and $Y^{(1, 1)} = \{y\}$,
$X^{(\omega)} = \{x\}$ and $Y^{(\omega)} = \{y\}$, and
$X^{\ll \omega, 1 \gg} = \{x\}$ and $Y^{\ll \omega, 1 \gg} = \{y\}$.

However we have

4.3.7 PROPOSITION: *X and Y are not l_p-equivalent.*

PROOF: Suppose $\phi: C_p(X) \to C_p(Y)$ is a linear homeomorphism. Let $\{W_n : n \in \mathbb{N}\}$ be a clopen decreasing base at y in Y such that for each $n \in \mathbb{N}$, $W_n \setminus W_{n+1}$ contains a clopen copy of $[1, \omega^\omega)$ and let $\{V_n : n \in \mathbb{N}\}$ be a clopen decreasing base at $\{x\} \cup \operatorname{supp}(y)$ in X. By corollary 4.3.5, there is $m \in \mathbb{N}$ such that

(V_m, W_m, m) is relatively bounded with respect to $(\{x\} \cup \operatorname{supp}(y), \{y\}, \phi)$,

and

(W_m, V_m, m) is relatively bounded with respect to $(\{y\}, \{x\} \cup \operatorname{supp}(y), \phi^{-1})$,

Notice that $X \setminus V_m$ is locally compact. So $X \setminus V_m = \bigcup_{i=1}^{\infty} A_i$ a clopen disjoint union such that for each $i \in \mathbb{N}$, A_i is compact. By corollary 1.2.15 (a) and lemma 1.2.10, there is $k \geq m$ such that

$$\operatorname{supp} W_k \cap \bigcup_{i>k} A_i = \emptyset.$$

Let $A = \bigcup_{i=1}^{k} A_i$. Then A is compact. Let K be a clopen copy of $[1, \omega^\omega)$ in $W_k \setminus W_{k+1}$. Write $K = \bigcup_{i=1}^{\infty} K_i$ as a clopen disjoint union such that for each $i \in \mathbb{N}$, $K_i = [1, \omega^i]$. Since $A \cup \operatorname{supp}(y)$ is compact, there is by corollary 1.2.15 (a) and lemma 1.2.10, $p > k$

such that

$$\operatorname{supp}(V_p \cup A \cup \operatorname{supp}(y)) \cap \bigcup_{i>p} K_i = \emptyset.$$

Note that the scattered height of $X \setminus V_p$ is less than ω. Let $s \in \mathbb{N}$ be the scattered height of $X \setminus V_p$.

Fix $i > p$. We have

$$\operatorname{supp}(A \cup V_p) \cap K_i = \emptyset, \text{ and}$$
$$\operatorname{supp} K_i \subset (A \cup V_p) \cup (V_m \setminus V_p).$$

Let $L_i = \overline{\operatorname{supp} K_i} \cap (V_m \setminus V_p)$. Then L_i is compact. If $L_i = \emptyset$, then $\operatorname{supp} K_i \subset A \cup V_p$, so by proposition 1.4.3,

$$K_i \subset \operatorname{supp} \operatorname{supp} K_i \subset \operatorname{supp}(A \cup V_p),$$

hence $K_i = \emptyset$. Contradiction. So L_i is a non-empty compactum.

By lemma 4.1.11, there is a continuous linear embedding $\eta_1 : C_0(K_i) \to C_0(Y)$ such that for each $f \in C(K_i)$ we have

$$\eta_1(f) \mid K_i = f,$$
$$\eta_1(f)(Y) \subset \operatorname{conv}(f(K_i) \cup \{0\}), \text{ and}$$
$$\eta_1(f)(Y \setminus K_i) = \{0\}.$$

Again by lemma 4.1.11 there is a continuous linear function $\eta_2 : C_0(K_i) \to C_0(X)$ such that for each $f \in C(L_i)$ we have

$$\eta_2(f) \mid L_i = f,$$
$$\eta_2(f)(X) \subset \operatorname{conv}(f(L_i) \cup \{0\}), \text{ and}$$
$$\eta_2(f)((X \setminus V_m) \cup V_p) = \{0\}.$$

Define

$\theta : C_0(K_i) \to C_0(L_i)$ by $\theta(f) = \phi^{-1}(\eta_1(f)) \mid L_i$, and
$\psi : C_0(L_i) \to C_0(K_i)$ by $\psi(f) = \phi(\eta_2(f)) \mid K_i$.

Then θ and ψ are clearly well-defined continuous linear functions.

CLAIM: θ is a linear m-embedding.

We first prove that θ is an embedding. It suffices to prove that for each $f \in C(K_i)$ we have $\psi(\theta(f)) = f$. Suppose there is $f \in C(K_i)$ such that $\psi(\theta(f)) \neq f$. Then

$$\phi(\eta_2(\theta(f))) \mid K_i \neq \eta_1(f) \mid K_i.$$

By corollary 1.4.2, we then have

$$\eta_2(\theta(\bigcup \cdot)) \mid \operatorname{supp} K_i \neq \phi^{-1}(\eta_1(f)) \mid \operatorname{supp} K_i.$$

Now $\eta_2(\theta(f)) \equiv 0$ on $(X \setminus V_m) \cup V_p$ and $\eta_1(f) \equiv 0$ on $X \setminus K_i$. Since $\operatorname{supp}(A \cup V_p) \subset X \setminus K_i$, it follows from corollary 1.4.2, that $\phi^{-1}(\eta_1(f)) \equiv 0$ on $A \cup V_p$. So

$$\eta_2(\theta(f)) \mid L_i \neq \phi^{-1}(\eta_1(f)) \mid L_i,$$

which implies that $\theta(f) \neq \theta(f)$, contradiction. We conclude that θ is a linear embedding.

To prove that θ is a linear m-embedding, first let $f \in C(K_i)$ be such that $f(K_i) \subset (-1/m, 1/m)$. Then

$$\eta_1(f)(Y) \subset \operatorname{conv}(f(K_i) \cup \{0\}) \subset (-1/m, 1/m), \text{ and}$$
$$\eta_1(f)((Y \setminus W_m) \cup \{y\}) = \{0\}.$$

Since $\operatorname{supp}(\{x\} \cup \operatorname{supp}(y)) \cap K_i = \emptyset$, we have

$$\eta_1(f)(\operatorname{supp}(\{x\} \cup \operatorname{supp}(y))) = \{0\},$$

so by corollary 1.4.2, $\phi^{-1}(\eta_1(f))(\{x\} \cup \operatorname{supp}(y)) = \{0\}$. Since (W_m, V_m, m) is relatively bounded with respect to $(\{y\}, \{x\} \cup \operatorname{supp}(y), \phi^{-1})$, we have $\phi^{-1}(\eta_1(f))(V_m) \subset (-1, 1)$. Since $L_i \subset V_m$, we have $\theta(f)(L_i) \subset (-1, 1)$.

Secondly let $f \in C(K_i)$ be such that $\theta(f)(L_i) \subset (-1, 1)$. Then

$$\eta_2(\theta(f))(X) \subset \operatorname{conv}(\theta(f)(L_i) \cup \{0\}) \subset (-1/m, 1/m), \text{ and}$$
$$\eta_2(\theta(f))((X \setminus V_m) \cup \{x\} \cup \operatorname{supp}(y)) = \{0\}.$$

Since $\eta_2(\theta(f))(\operatorname{supp}(y)) = \{0\}$, we have by corollary 1.4.2, $\phi(\eta_2(\theta(f)))(y) = 0$. Since (V_m, W_m, m) is relatively bounded with respect to $(\{x\} \cup \operatorname{supp}(y), \{y\}, \phi)$, we have $\phi(\eta_2(\theta(f)))(W_m) \subset (-1, 1)$. Since $K_i \subset W_m$, we have $\psi(\theta(f))(K_i) \subset (-1, 1)$. By the above $\psi(\theta(f)) = f$, so $f(K_i) \subset (-1, 1)$. We conclude that θ is a linear m-embedding.

Since $L_i \subset X \setminus V_p$ and $\kappa(X \setminus V_p) = s$, there is a linear 1-embedding from $C_0(L_i)$ into $C_0([1, \omega^{s+1}])$. By the claim we have for each $i \in \mathbb{N}$ a linear m-embedding from $C_0([1, \omega^i])$ into $C_0([1, \omega^{s+1}])$. then by lemma 4.3.6 we have a linear embedding from $C_0([1, \omega^\omega])$ into $C_0([1, \omega^{s+2}])$. This is a contradiction with lemma 2.6.7. We conclude that X and Y are not l_p-equivalent. □

If we look at the spaces X and Y we see that each neighborhood of $y \in Y$ contains a closed copy of $[1, \omega^\omega)$ and that no neighborhood of $x \in X$ contains a closed copy of $[1, \omega^\omega)$.

For a space X, let

$$X^{<\alpha>} = \{x \in X : \text{each neighborhood of } x \text{ contains a closed copy of } [1, \omega^\alpha)\}.$$

Obviously $X^{<\alpha>}$ is a closed subspace of X. If $X^{<\alpha>} = \emptyset$, then $X^{(\alpha)}$ is locally compact. Indeed if $x \in X^{(\alpha)}$, then each neighborhood contains a closed copy of $[1, \omega^\alpha] \times \mathbb{N}$ (cf. lemma 4.1.8). But then each neighborhood also contains a closed copy of $[1, \omega^\alpha)$.

Let $\alpha \geq \omega$ be a countable prime component and let β be a countable ordinal. By $X^{<\alpha, \beta>}$ we denote the set $X^{\{\beta\}}$ with respect to the pair $(X, X^{<\alpha>})$. Notice that if β is a successor, say $\beta = \gamma + 1$, then we have $X^{<\alpha, \beta>} = (X^{<\alpha, \gamma>})^{(0, 1)}$.

The numbers $X<\alpha, \beta>$ are defined for ordinals α and β similarly to the numbers $X(\alpha, \beta)$ as follows:

$X<\alpha, \beta> = 0$ if and only if $X^{<\alpha, \beta>} = \emptyset$,
$X<\alpha, \beta> = 1$ if and only if $X^{<\alpha, \beta>}$ is non-empty and compact, and
$X<\alpha, \beta> = 2$ if and only if $X^{<\alpha, \beta>}$ is not compact.

Question 2: Let $\alpha \geq \omega$ be a countable prime component and let X and Y be l_p-equivalent separable metric zero-dimensional spaces. Are $(X, X^{<\alpha>})$ and $(Y, Y^{<\alpha>})$ l_p-equivalent pairs?

We conjecture that this question has a positive answer.

§4.4. Remarks on a conjecture

In sections 4.1 and 4.3 we found several l_p-equivalent pairs. In this section we conjecture that these l_p-equivalent pairs together with the conjectured l_p-equivalent pairs in question 1 and 2 in section 4.3 are sufficient to obtain an isomorphical classification for the function spaces $C_p(X)$, for countable metric X. In this section we indicate among other things difficulties that one encounters if one tries to prove the conjecture along the lines of the proof of theorem 4.2.18.

4.4.1 CONJECTURE: *Let X and Y be infinite countable metric spaces. Then X and Y are l_p-equivalent if and only if for all countable ordinals α, β and γ, where α is a prime component, we have*

(*a*) $X(\alpha, \beta) = Y(\alpha, \beta)$,

(*b*) $X<\alpha, \beta> = Y<\alpha, \beta>$, *for* $\alpha \geq \omega$, *and*
(*c*) $X\ll\alpha, \beta, \gamma\gg = Y\ll\alpha, \beta, \gamma\gg$, *for* $\alpha \geq \omega$.

Note that conditions (b) and (c) are trivially true when X and Y are countable metric spaces with scattered height less than or equal to ω, so the conjecture is in agreement with theorem 4.2.18 when we restrict ourselves to this class of spaces.

Part of the proof of theorem 4.2.18 depends in a rough way on lemma 4.2.10 and on the decomposition theorems at the beginning of section 4.2. If we want to prove for countable metric spaces satisfying conditions (a), (b) and (c) in the above conjecture, that they are l_p-equivalent, and we want to follow the same strategy as in section 4.2, some problems appear. For example, it is not clear how to decompose each countable metric space such that lemma 4.2.10 becomes applicable. Such a decomposition should have the property that between "building blocks" we have "linear k-homeomorphisms". A second problem is how we should start an inductive proof such as in the proof of lemma 4.2.13. We will try to show why these problems are real obstacles in the process of proving the conjecture. We first present a lemma which gives connections between the operations that are involved in the conjecture. One could say that this lemma is the "replacement" of lemma 4.2.1.

4.4.2 LEMMA: *Let X be a space. Let $\alpha, \alpha_1, \alpha_2, \beta_1$ and β_2 be ordinals with α, α_1 and α_2 prime components. Then*
(*a*) $X^{(\alpha, 1)} \subset X^{<\alpha>} \subset X^{(\alpha)}$,
(*b*) $X^{(\alpha_1, \alpha_2-\alpha_1)} \subset X^{(\alpha_2)} \subset X^{(\alpha_1)}$ $(\alpha_1 \leq \alpha_2)$,
(*c*) $X^{\ll\alpha, \beta_1, \beta_2-\beta_1\gg} \subset X^{\ll\alpha, \beta_2\gg} \subset X^{\ll\alpha, \beta_1\gg}$ $(\beta_1 \leq \beta_2, \alpha \geq \omega)$, *and*
(*d*) $X^{<\alpha>} \subset X^{\ll\alpha, 1\gg}$ $(\alpha \geq \omega)$.

PROOF: For (a) let $x \in X^{(\alpha, 1)}$. Then each neighborhood of x in X contains a closed copy of $[1, \omega^\alpha] \times \mathbb{N}$, hence a closed copy of $[1, \omega^\alpha)$. So $x \in X^{<\alpha>}$. For $x \in X^{<\alpha>}$ each neighborhood of x contains a closed copy of $[1, \omega^\alpha)$, hence $x \in X^{(\alpha)}$.

For (b) we have by proposition 2.2.2, $X^{(\alpha_2)} \subset X^{(\alpha_1)}$. Furthermore

$$X^{(\alpha_1, \alpha_2-\alpha_1)} \subset (X^{(\alpha_1)})^{(\alpha_2-\alpha_1)} = (X^{(\alpha_1+(\alpha_2-\alpha_1))}) = X^{(\alpha_2)}.$$

This completes the proof of (b).

For (c), observe that the second inclusion is a triviality, so we only have to prove that for $\delta < \alpha$, $X^{\ll\alpha, \beta_1, \beta_2-\beta_1\gg} \subset X^{(\delta, \beta_2)}$. We will prove this by transfinite induction on β_2. If $\beta_2 = 0$, $\beta_1 = 0$ and we are done. Suppose for all $\beta < \beta_2$, the inclusion is true. If $\beta_1 = \beta_2$ there is nothing to prove, so suppose $\beta_1 < \beta_2$. If β_2 is a successor, say

$\beta_2 = \beta + 1$, we have $\beta_2 - \beta_1 = (\beta - \beta_1) + 1$, so

$$X^{\ll\alpha, \beta_1, \beta_2-\beta_1\gg} = (X^{\ll\alpha, \beta_1, \beta-\beta_1\gg})^{(0, 1)} \subset (X^{(\delta, \beta)})^{(0, 1)} \subset X^{(\delta, \beta_2)}.$$

If β is a limit ordinal we have

$$X^{\ll\alpha, \beta_1, \beta_2-\beta_1\gg} = \bigcap_{\beta_1<\gamma<\beta_2} X^{\ll\alpha, \beta_1, \gamma-\beta_1\gg} = \bigcap_{\beta_1<\gamma<\beta_2} X^{(\delta, \gamma)} \subset X^{(\delta, \beta_2)}.$$

This completes the proof of case (c).

For (d) we have to prove for $\delta < \alpha$ that $X^{<\alpha>} \subset X^{(\delta, 1)}$. So let $x \in X^{<\alpha>}$. Then each neighborhood U_x of x contains a closed copy of $[1, \omega^\alpha)$. Hence $U_x^{(\delta)}$, cannot be compact, so $x \in X^{(\delta, 1)}$. This completes the proof of (e) and hence the proof of this lemma. □

4.4.3 COROLLARY: *Let X be a space. Let $\alpha, \alpha_1, \alpha_2, \beta, \beta_1$ and β_2 be ordinals with α, α_1 and α_2 prime components. Then*

(a) $X^{(\alpha, 1+\beta)} \subset X^{<\alpha, \beta>} \subset X^{(\alpha, \beta)}$,

(b) $X^{(\alpha_1, \alpha_2-\alpha_1+\beta)} \subset X^{(\alpha_2, \beta)} \subset X^{(\alpha_1, \beta)}$ $(\alpha_1 \leq \alpha_2)$,

(c) $X^{\ll\alpha, \beta_1, \beta_2-\beta_1+\beta\gg} \subset X^{\ll\alpha, \beta_2, \beta\gg} \subset X^{\ll\alpha, \beta_1, \beta\gg}$ $(\beta_1 \leq \beta_2, \alpha \geq \omega)$, *and*

(d) $X^{<\alpha, \beta>} \subset X^{\ll\alpha, 1, \beta\gg}$ $(\alpha \geq \omega)$.

From the lemma it also follows that some of the inclusions are in fact equalities. For example we have the following

4.4.4 COROLLARY: *Let X be a space and let α, β be countable ordinals with α a prime component and $\beta \geq \omega$. Then $X^{(\alpha, \beta)} = X^{<\alpha, \beta>}$.*

PROOF: This follows directly from corollary 4.4.3 (a). □

Comparing corollary 4.4.3 with lemma 4.2.1, we see that the general situation (countable metric spaces) gives rise to many more cases than the specific situation in section 4.2 (countable metric spaces with scattered height less than or equal to ω). Our next task is to find substitutes for corollary 4.2.2 and lemma 4.2.3. This is almost impossible. To make this clear we will restrict ourselves from now on to a specific class of spaces, which is the most natural one to consider after the results in section 4.2. We will consider countable metric spaces X with $\omega < \kappa(X) < \omega^2$. Many of the conditions in our conjecture become empty in this situation. The only ordinals we have to consider are $\alpha = 0$, $\alpha = 1$, $\alpha = \omega$ and $\beta, \gamma < \omega^2$. If we reformulate corollary 4.4.3 we get

4.4.5 LEMMA: *Let X be a space. Let $\beta, \beta_1, \beta_2 < \omega^2$ be ordinals. Then*

(a) $X^{(\omega, 1+\beta)} \subset X^{<\omega, \beta>} \subset X^{(\omega, \beta)}$,

(b) *(i)* $X^{(0, 1+\beta)} \subset X^{(1, \beta)} \subset X^{(0, \beta)}$,

(ii) $X^{(0, \omega+\beta)} \subset X^{(\omega, \beta)} \subset X^{(0, \beta)}$,

(iii) $X^{(1, \omega+\beta)} \subset X^{(\omega, \beta)} \subset X^{(1, \beta)}$,

(c) $X^{\ll\omega, \beta_1, \beta_2-\beta_1+\beta\gg} \subset X^{\ll\omega, \beta_2, \beta\gg} \subset X^{\ll\omega, \beta_1, \beta\gg}$ $(\beta_1 \leq \beta_2)$, *and*

(d) $X^{<\omega, \beta>} \subset X^{\ll\omega, 1, \beta\gg}$. □

In analogy with corollary 4.2.2 we define

$$\pi_0 = \min\{\pi : X(0, \pi) = 0\},$$
$$\pi_1 = \min\{\pi : X(1, \pi) = 0\},$$
$$\gamma_\omega = \min\{\gamma : X<\omega, \gamma> = 0\},$$
$$\pi_\omega = \min\{\pi : X\ll\omega, \pi, 0\gg = 0\},$$

and for every $\pi \leq \pi_\omega$,

$$\beta_\pi = \min\{\beta : X\ll\omega, \pi, \beta\gg = 0\}.$$

Of course all these ordinals are well-defined and less than or equal to $\kappa(X)$. Note that $\pi_\omega > 0$ because of our choice of X. So β_1 is well-defined.

4.4.6 LEMMA: *We have the following relations*

(a) $\pi_1 \leq \pi_0 \leq 1 + \pi_1$,

(b) $\gamma_\omega \leq \beta_1 \leq \beta_0 \leq 1 + \gamma_\omega \leq \pi_\omega \leq \pi_1 \leq \pi_0 \leq \omega + \beta_0$,

(c) $0 = \beta_{\pi_\omega} \leq \beta_\tau \leq \beta_\sigma \leq \tau - \sigma + \beta_\tau$ $(\sigma \leq \tau \leq \pi_\omega)$, *and*

(d) *for all* $\pi < \pi_\omega$, $\beta_\pi \geq 1$.

PROOF: As in corollary 4.2.2, part (a) follows from lemma 4.4.5 (b)(i). For (b), notice that by lemma 4.4.5 (d),

$$X^{<\omega, \beta_1>} \subset X^{\ll\omega, 1, \beta_1\gg} = \emptyset,$$

hence $\gamma_\omega \leq \beta_1$. That $\beta_1 \leq \beta_0$ follows from (c) and will be proved there. By lemma 4.4.5 (a), we have

$$X^{\ll\omega, 0, 1+\gamma_\omega\gg} = X^{(\omega, 1+\gamma_\omega)} \subset X^{<\omega, \gamma_\omega>} = \emptyset,$$

hence $\beta_0 \leq 1 + \gamma_\omega$. Since $\pi_\omega > o$, we have by lemma 4.4.5 (c) and (d),

$$X^{<\omega, \pi_\omega - 1>} \subset X^{\ll\omega, 1, \pi_\omega - 1\gg} \subset X^{\ll\omega, \pi_\omega\gg} = \emptyset,$$

hence $\gamma_\omega \leq \pi_\omega - 1$. This implies $1 + \gamma_\omega \leq \pi_\omega$. Since

$$X^{\ll \omega, \pi_1 \gg} = \bigcap_{n<\omega} X^{(n, \pi_1)} \subset X^{(1, \pi_1)} = \emptyset,$$

we have $\pi_\omega \leq \pi_1$. That $\pi_1 \leq \pi_0$ was proved under (a). Finally by lemma 4.4.5 (b)(ii) we have

$$X^{(0, \omega+\beta_0)} \subset X^{(\omega, \beta_0)} = X^{\ll \omega, 0, \beta_0 \gg} = \emptyset,$$

hence $\pi_0 \leq \omega + \beta_0$. This completes the proof of (b).

For (c), we have by lemma 4.4.5 (c),

$$X^{\ll \omega, \sigma, \tau - \sigma + \beta_\tau \gg} \subset X^{\ll \omega, \tau, \beta_\tau \gg} = \emptyset,$$

hence $\beta_\sigma \leq \tau - \sigma + \beta_\tau$, and

$$X^{\ll \omega, \tau, \beta_\tau \gg} \subset X^{\ll \omega, \sigma, \beta_\sigma \gg} = \emptyset,$$

hence $\beta_\tau \leq \beta_\sigma$. Obviously $\beta_{\pi_\omega} = 0$. This proves (c).

Part (d) follows immediately by the definition of π_ω. This completes the proof of this lemma. □.

4.4.7 COROLLARY: *If* $\sigma < \pi_\omega$, *then*

(*a*) $\beta_{\sigma+1} \leq \beta_\sigma \leq 1 + \beta_{\sigma+1}$, *and*

(*b*) $\beta_\sigma \leq \pi_\omega - \sigma$.

Furthermore we have $\beta_0 \geq 1$, *and for* $\pi \geq \pi_0$, $\beta_\pi = 0$.

PROOF: Part (a) and (b) are special cases of lemma 4.4.6 (c). By lemma 4.4.6 (b) we have $\pi_\omega \geq 1 + \gamma_\omega \geq 1 > 0$, hence by lemma 4.4.6 (d), $\beta_0 \geq 1$. Since $\pi_\omega \leq \pi_0$, we have for $\pi \geq \pi_0$, $\beta_\pi = 0$. □

Comparing lemma 4.4.6 with corollary 4.2.2 we notice that lemma 4.4.6 covers more cases than corollary 4.2.2. If we want to follow the same strategy as in section 4.2, our next step should be to find a substitute for lemma 4.2.3. In this lemma we had for each natural number exactly six possible cases. This made the situation there suitable for the inductive proof in lemma 4.2.13. The role of that natural number is now played by π_0. However if we increase π_0, the number of possibilities will also increase. This makes it difficult to find a suitable replacement for lemma 4.2.3. To make this clear we will now look at some specific values of π_0. If π_0 is fixed, γ_ω, π_ω, π_1 and β_π for $\pi < \pi_0$ are the only defined ordinals that can possibly be non-zero (cf. lemma 4.4.6 and corollary 4.4.7).

First let $\pi_0 = 1$. Then we have to consider γ_ω, π_ω, π_1 and β_0. By lemma 4.4.6 (b) and corollary 4.4.7 we then have $1 = \beta_0 \leq 1 + \gamma_\omega \leq \pi_\omega \leq \pi_1 \leq \pi_0$. This gives

$\pi_0 = 1$			
γ_ω	β_0	π_ω	π_1
0	1	1	1

A space X satisfying this condition can be one of the following types (cf. lemma 4.2.3).

(1) $X(0, 0) = X(1, 0) = X(\omega, 0) = 1$,
(2) $X(0, 0) = 2$, and $X(1, 0) = X(\omega, 0) = 1$,
(3) $X(0, 0) = X(1, 0) = 2$, and $X(\omega, 0) = 1$, and
(4) $X(0, 0) = X(1, 0) = X(\omega, 0) = 2$,

Spaces of each of these four types in fact exist. We will give the examples, but leave all calculations to the reader. For case (1) let $X = [1, \omega^\omega]$, and for case (2) let $X = [1, \omega^\omega] \oplus \mathbb{N}$. The space $[1, \omega^\omega] \oplus ([1, \omega] \times \mathbb{N})$ satisfies the conditions of (3) and $[1, \omega^\omega] \times \mathbb{N}$ satisfies the conditions of (4).

Let us now consider the case that $\pi_0 = 2$. We then have to consider γ_ω, π_ω, π_1, β_1 and β_0. Lemma 4.4.6 and corollary 4.4.7 give us the following possibilities

	$\pi_0 = 2$				
	γ_ω	β_1	β_0	π_ω	π_1
1	0	0	1	1	1
2	0	0	1	1	2
3	0	1	1	2	2
4	1	1	1	2	2
5	1	1	2	2	2

Each possibility gives rise to several cases in the same way as for $\pi_0 = 1$. The number of these possibilities increases since there is one more variable. We will not try to explicitly describe all these cases, but instead we shall give one example for each of the five cases in the above table, to make clear that the relations in lemma 4.4.6 are at

least in this case sharp.

A space satisfying the conditions in (1) is $T \oplus [1, \omega^\omega]$ and a space satisfying the conditions in (2) is $T([1, \omega]) \oplus [1, \omega^\omega]$. Let X be the space obtained from T by replacing each $(i, j) \in T$ by a copy of $[1, \omega^j]$. Then X satisfies the conditions in (3). For case (4) we can take $X = S_1([1, \omega^\omega))$ and for case (5) we can take $X = T([1, \omega^\omega])$. Again we leave all calculations to the reader.

One can see that for $\pi_0 = 2$, the situation involves a lot more possibilities then for $\pi_0 = 1$. If we consider $\pi_0 = 3$, we get another new variable β_2, and by lemma 4.4.6 and corollary 4.4.7 the following table:

$\pi_0 = 3$						
	γ_ω	β_2	β_1	β_0	π_ω	π_1
1	0	0	0	1	1	2
2	0	0	1	1	2	2
3	0	0	0	1	1	3
4	0	0	1	1	2	3
5	0	1	1	1	3	3
6	1	0	1	1	2	2
7	1	0	1	2	2	2
8	1	0	1	1	2	3
9	1	0	1	2	2	3
10	1	1	1	1	3	3
11	1	1	1	2	3	3
12	1	1	2	2	3	3
13	2	1	2	2	3	3
14	2	1	2	3	3	3

Hence $\pi_0 = 3$ yields 14 possibilities and each of them can be dealt with as in the proof of lemma 4.2.3. It is also possible to describe for each of the above cases an example of a space satisfying the corresponding conditions. It goes too far to present them here.

For $\pi_0 = 4$, the number of cases has increased to 32, and things get even worse if π_0

is infinite. Then there are infinitely many variables to consider, and the number of possibilities will also be infinite. But if we want to follow the same strategy as in section 4.2, we need an explicit description of all possibilities and that is exactly the problem when one wants to find "decomposition lemmas" which can be used in a lemma such as lemma 4.2.13. It is also not obvious that the relations in lemma 4.4.6 are sharp. Can one find for each possibility a space satisfying the corresponding condition?

Another problem in connection with lemma 4.2.13 is to find "linear k-homeomorphisms". In section 4.2 we were in the pleasant situation that for each pair of l_p-equivalent spaces both with scattered height less than ω, we actually had a linear k-homeomorphism for some $k \in \mathbb{N}$. That property is lost if we consider spaces X with $\omega < \kappa(X) < \omega^2$. This problem already occurs in the case of spaces satisfying $\pi_0 = 1$. These spaces should be the "building blocks" for spaces with $\pi_0 = 2$ (if we want to start a proof by induction on π_0). If we let $X = [1, \omega^\omega] \oplus ([1, \omega] \times \mathbb{N})$ and $Y = [1, \omega^\omega] \oplus [1, \omega^\omega)$, then both spaces have scattered height $\omega + 1$, and they both have $\pi_0 = 1$. By theorem 2.6.17 they are moreover l_p-equivalent. However, we cannot use X and Y as building blocks for obtaining linear homeomorphisms between function spaces of spaces with $\pi_0 = 2$ using lemma 4.2.10 because of the following

4.4.8 LEMMA: *There is no linear k-homeomorphism between $C_p(X)$ and $C_p(Y)$ for any $k \in \mathbb{N}$.*

PROOF: Suppose there is $k \in \mathbb{N}$ and a linear k-homeomorphism between $C_p(X)$ and $C_p(Y)$. Then by lemma 4.2.10, $S_1(X)$ and $S_1(Y)$ are l_p-equivalent. However $(S_1(X))^{\ll \omega, 1 \gg} = \emptyset$ and $(S_1(Y))^{\ll \omega, 1 \gg} \neq \emptyset$. This contradicts corollary 4.3.2. □

Comparing the situation here to the one in section 4.2, we see that if we want to "decompose" spaces into spaces of a "lower level", we are forced to avoid situations as above in order to make it possible to apply lemma 4.2.10.

Summarizing we conclude that a proof of the conjecture, even in the case of the relatively simple spaces X with $\omega < \kappa(X) < \omega^2$, will be a hard job and will certainly not be as "simple" as the proof given in section 4.2.

We will finish this section by giving a relatively simple result concerning the l_p-equivalence of non-scattered countable metric spaces.

4.4.9 PROPOSITION: *Let X be a non-scattered countable metric space. Then X and $\mathbb{Q}$ are l_p-equivalent.*

PROOF: Since X is a non-scattered countable metric space, X contains a closed copy D of $\mathbb{Q}$. Since X is a countable metric space, $\mathbb{Q}$ contains a closed copy E of $X \times \mathbb{N}$. Then

$$\begin{aligned}
C_p(X) &\sim C_{p,D}(X) \times C_p(\mathbb{Q}) && \text{by proposition 2.3.2} \\
&\sim C_{p,D}(X) \times C_p(\mathbb{Q}) \times C_p(\mathbb{Q}) && \mathbb{Q} \approx \mathbb{Q} \oplus \mathbb{Q} \\
&\sim C_p(X) \times C_{p,E}(\mathbb{Q}) \times \Pi_{i=1}^{\infty} C_p(X) && \text{by proposition 2.3.2} \\
&\sim C_{p,E}(\mathbb{Q}) \times \Pi_{i=1}^{\infty} C_p(X) && \\
&\sim C_p(\mathbb{Q}) && \text{by proposition 2.3.2 } \square
\end{aligned}$$

§4.5. Partial results on l_0-equivalence

In the previous section we saw that a complete isomorphical classification for the function spaces $C_p(X)$, for X countable and metric, seems beyond reach. Only for countable metric spaces with scattered height less than or equal to ω the situation is clear. In this section we deal with the compact-open topology instead of the topology of pointwise convergence and we try to make clear that an isomorphical classification for the function spaces $C_0(X)$, where X is countable and metric, seems even more beyond reach. First we notice that from theorem 4.2.18 and corollary 1.2.21 we have the following

4.5.1 THEOREM: *Let X and Y be infinite countable metric spaces, such that $\kappa(X), \kappa(Y) \leq \omega$, and for every $n \in \mathbb{N}$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$. Then X and Y are l_0-equivalent.* $\square$

In the proof of the converse implication for l_p-equivalence, we used the notion of l_p-equivalent pairs. Of course we can define in a similar way the notion of l_0-equivalent pairs, and it is then possible to prove a theorem such as theorem 4.1.7. The problem however lies in propositions 4.1.9 and 4.1.12, where we used the precise description of supports obtained in section 1.4 for the topology of pointwise convergence. We were unable to derive such a precise description of supports in the case of the compact-open topology. This complicates the situation quite a bit. We will now derive two theorems that should be compared with corollary 4.1.14.

4.5.2 THEOREM: *Let X and Y be zero-dimensional separable metric l_0-equivalent spaces. Then*

(a) $X=\emptyset$ if and only if $Y=\emptyset$,

(b) X is compact if and only if Y is compact, and

(c) X is locally compact if and only if Y is locally compact.

PROOF: Part (a) is a triviality. Part (b) follows from theorem 1.5.7. For (c) let $\phi: C_0(X)\to C_0(Y)$ be a linear homeomorphism and suppose that X is locally compact and Y is not locally compact. Then X can be written as a clopen disjoint union $X=\bigcup_{i=1}^{\infty}A_i$ such that for each $i\in\mathbb{N}$, A_i is compact. Let $y\in Y$ be such that y has no compact neighborhood and let $\{U_n : n\in\mathbb{N}\}$ be a decreasing clopen base at y. By corollary 1.2.15 (a) and lemma 1.2.10, there is $n\in\mathbb{N}$ such that

$$(\operatorname{supp} U_n)\cap\bigcup_{i>n}A_i=\emptyset.$$

Let $A=\bigcup_{i=1}^{n}A_i$. Then A is compact. So $\overline{\operatorname{supp} A}$ is compact as well. Since U_n is not compact, there is a non-empty clopen $O\subset U_n\setminus\overline{\operatorname{supp} A}$. Now let $f\neq 0$ be a Urysohn function such that $f(Y\setminus O)=\{0\}$. Since $Y\setminus O$ is a neighborhood of $\overline{\operatorname{supp} A}$, by corollary 1.2.15 (a), $\phi^{-1}(f)(A)=\{0\}$. Since A is a neighborhood of $\operatorname{supp} O$, we consequently have $f(O)=\{0\}$. But then $f=0$, which is a contradiction. This proves the theorem. □

4.5.3 THEOREM: *Let X and Y be zero-dimensional separable metric l_0-equivalent spaces. Then*

(a) $X^{(1)}=\emptyset$ if and only if $Y^{(1)}=\emptyset$,

(b) $X^{(1)}$ is compact if and only if $Y^{(1)}$ is compact, and

(c) $X^{(1)}$ is locally compact if and only if $Y^{(1)}$ is locally compact.

PROOF: Let $\phi: C_0(X)\to C_0(Y)$ be a linear homeomorphism. For (a) suppose $X^{(1)}=\emptyset$ and $Y^{(1)}\neq\emptyset$. Let K be a copy of $[1,\omega]$ in Y and let $L=\operatorname{supp} K$. Then L is non-empty and compact and hence is finite. By lemma 4.1.11, there is a continuous linear function $\eta_1: C_0(K)\to C_0(Y)$ such that for each $f\in C(K)$, $\eta_1(f)\mid K=f$ and there is a continuous linear function $\eta_2: C_0(L)\to C_0(X)$ such that for each $f\in C(L)$, $\eta_2(f)\mid L=f$. Define

$\theta: C_0(K)\to C_0(L)$ by $\theta(f)=\phi^{-1}(\eta_1(f))\mid L$, and
$\psi: C_0(L)\to C_0(K)$ by $\psi(g)=\phi(\eta_2(g))\mid K$.

CLAIM: For every $f\in C(K)$, $\psi(\theta(f))=f$.

To the contrary suppose there is $f\in C(K)$ such that $\psi(\theta(f))\neq f$. Then

$\phi(\eta_2(\theta(f)) \restriction K \neq \eta_1(f) \restriction K$. Since L is open, L is a neighborhood of $\operatorname{supp} K$. Hence by corollary 1.2.15 (a), $\eta_2(\theta(f)) \restriction L \neq \phi^{-1}(\eta_1(f)) \restriction L$, which gives $\theta(f) \neq \theta(f)$. Contradiction. This proves the claim.

We conclude that θ is a linear embedding. So we have a linear embedding from $C_0([1, \omega])$ into some $\mathbb{R}^n$ for $n \in \mathbb{N}$. This is not possible since the algebraic dimension of $C_0([1, \omega])$ is infinite. This proves (a).

For (b) suppose $X^{(1)}$ is compact and $Y^{(1)}$ is not compact. By (a) we have $X^{(1)} \neq \emptyset$. Let $\{y_n : n \in \mathbb{N}\}$ be a closed discrete subset of Y consisting of non-isolated points. Let $\{O_n : n \in \mathbb{N}\}$ be a clopen discrete family such that for each $n \in \mathbb{N}$, $y_n \in O_n$. Let $\{U_n : n \in \mathbb{N}\}$ be a clopen decreasing base at $X^{(1)}$ in X. By corollary 1.2.15 (a) and lemma 1.2.10, there is $k \in \mathbb{N}$ such that

$$\operatorname{supp} U_k \cap \bigcup_{i \geq k} O_i = \emptyset.$$

Find a copy K of $[1, \omega]$ in O_k containing y_k. Let $L = \overline{\operatorname{supp} K} \cap X \setminus U_k$. Then L is compact, and hence is finite. If $L = \emptyset$, then U_k is a neighborhood of $\operatorname{supp} K$. Furthermore $Y \setminus O_k$ is a neighborhood of $\operatorname{supp} U_k$. Let f be a Urysohn function such that $f(y_k) = 1$ and $f(Y \setminus O_k) = \{0\}$. Since ϕ^{-1} is effective, $\phi^{-1}(f)(U_k) = \{0\}$. By effectiveness of ϕ, we then have $f(K) = \{0\}$. But this gives a contradiction since $y_k \in K$. We conclude that $L \neq \emptyset$. By lemma 4.1.11 and proposition 1.2.19, there is a continuous linear function $\eta_1 : C_0(K) \to C_0(Y)$ such that for each $f \in C(K)$, $\eta_1(f) \restriction K = f$ and $\eta_1(f)(Y \setminus O_k) = \{0\}$, and there is a continuous linear function $\eta_2 : C_0(L) \to C_0(X)$ such that for each $f \in C(L)$, $\eta_2(f) \restriction L = f$ and $\eta_2(f)(U_k) = \{0\}$. Define $\theta : C_0(K) \to C_0(L)$ by $\theta(f) = \phi^{-1}(\eta_1(f)) \restriction L$, and $\psi : C_0(L) \to C_0(K)$ by $\psi(g) = \phi(\eta_2(g)) \restriction K$.

CLAIM: For every $f \in C(K)$, $\psi(\theta(f)) = f$.

To the contrary suppose there is $f \in C_0(K)$ such that $\psi(\theta(f)) \neq f$. Then $\phi(\eta_2(\theta(f)) \restriction K \neq \eta_1(f) \restriction K$. Since $U_k \cup L$ is a neighborhood of L, we have by effectiveness of ϕ that $\eta_2(\theta(f)) \restriction (U_k \cup L) \neq \phi^{-1}(\eta_1(f)) \restriction (U_k \cup L)$. Now $\eta_2(\theta(f)) \equiv 0$ on U_k and $\eta_1(f) \equiv 0$ on $Y \setminus O_k$. Since $Y \setminus O_k$ is a neighborhood of $\operatorname{supp} U_k$ we have by effectiveness of ϕ^{-1}, $\phi^{-1}(\eta_1(f)) \equiv 0$ on U_k. We conclude that $\eta_2(\theta(f)) \restriction L \neq \phi^{-1}(\eta_1(f) \restriction L$, hence $\theta(f) \neq \theta(f)$. Contradiction. This proves the claim.

From the claim it follows that θ is a linear embedding. Again we have a linear embedding from $C_0([1, \omega])$ into some $\mathbb{R}^n$ for $n \in \mathbb{N}$, which gives a contradiction. This proves case (b).

For (c) suppose that $X^{(1)}$ is locally compact and $Y^{(1)}$ is not locally compact. By (b) we have that $X^{(1)}$ is not compact. Hence X can be written as a clopen disjoint union

$X = \bigcup_{i=1}^{\infty} A_i$ such that for each $i \in \mathbb{N}$, $A_i^{(1)}$ is a non-empty compactum. Let $y \in Y^{(1)}$ be a point and let $\{U_n : n \in \mathbb{N}\}$ be a clopen decreasing base at y such that for each $n \in \mathbb{N}$, $U_n^{(1)}$ is not compact. By corollary 1.2.15 (a) and lemma 1.2.10, there is $k \in \mathbb{N}$ such that

$$\operatorname{supp} U_k \cap \bigcup_{i>k} A_i = \emptyset.$$

Let $A = \bigcup_{i=1}^{k} A_i$. Then $A^{(1)}$ is a non-empty compactum. Let $\{V_n : n \in \mathbb{N}\}$ be a clopen decreasing base at $A^{(1)}$ in A. Let $\{y_n : n \in \mathbb{N}\}$ be a closed discrete set of non-isolated points in U_k and let $\{O_n : n \in \mathbb{N}\}$ be a clopen discrete family in U_k such that for each $n \in \mathbb{N}$, $y_n \in O_n$. By corollary 1.2.15 (a) and lemma 1.2.10, there is $p \in \mathbb{N}$ such that

$$\operatorname{supp} V_p \cap \bigcup_{i \geq p} O_i = \emptyset.$$

Find a copy K of $[1, \omega]$ in O_k containing y_k. Let $L = \overline{\operatorname{supp} K} \cap X \setminus V_p$. As under case (b) we can derive a contradiction. This proves the theorem. □

In the above theorems the proofs are similar to the proofs in section 4.1. The next prime component to consider is ω. The specific problems that we encounter when dealing with the compact-open topology now become clear. We are able to prove the following theorem which is much weaker than the result we have for l_p-equivalence. It is a generalization of a result in [6].

4.5.4 THEOREM: *Let X and Y be zero-dimensional separable metric spaces. Suppose $X^{(1)}$ is discrete and $Y^{(\omega)} \neq \emptyset$. Then X and Y are not l_0-equivalent.*

PROOF: To the contrary suppose there is a linear homeomorphism $\phi: C_0(X) \to C_0(Y)$. Write X as a clopen disjoint union $X = \bigcup_{i=1}^{\infty} X_i$ such that for each $i \in \mathbb{N}$, $X_i^{(1)}$ contains at most one point. By theorem 4.5.3, there is at least one $i \in \mathbb{N}$ with $X_i^{(1)} \neq \emptyset$. Let K be a copy of $[1, \omega^{\omega}]$ in Y and let $L = \overline{\operatorname{supp} K}$. Then L is compact, hence there is $p \in \mathbb{N}$ such that $L \subset \bigcup_{i=1}^{p} X_i$. We may assume that for each $i \in \mathbb{N}$, $X_i^{(1)} \neq \emptyset$, and we let x_i denote the unique point in $X_i^{(1)}$.

CLAIM 1: For every $f, g \in C_0(Y)$ with $f \mid K \neq g \mid K$ it follows that $\phi^{-1}(f) \mid L \neq \phi^{-1}(g) \mid L$.

Let $y \in K$ be such that $f(y) \neq g(y)$ and let W_0 and W_1 be disjoint open neighborhoods of $f(y)$ and $g(y)$ in $\mathbb{R}$, respectively. Then $A(\{y\}, W_0)$ and $A(\{y\}, W_1)$ are disjoint open neighborhoods of f and g in $C_0(Y)$. So $\phi^{-1}(A(\{y\}, W_0))$ and $\phi^{-1}(A(\{y\}, W_1))$ are disjoint open neighborhoods of $\phi^{-1}(f)$ and $\phi^{-1}(g)$ in $C_0(X)$.

There consequently exist compact subsets $K_1, \ldots, K_n, L_1, \ldots, L_m$ of X and open subsets $U_1, \ldots, U_n, V_1, \ldots, V_m$ of $\mathbb{R}$ such that

$$\phi^{-1}(f) \in \bigcap_{i=1}^{n} A(K_i, U_i) \subset \phi^{-1}(A(\{y\}, W_0))$$

and

$$\phi^{-1}(g) \in \bigcap_{i=1}^{m} A(L_i, V_i) \subset \phi^{-1}(A(\{y\}, W_1)).$$

We claim there is a $z \in \operatorname{supp}(y) \subset L$ such that $\phi^{-1}(f)(z) \neq \phi^{-1}(g)(z)$ (and then we are done). Striving for a contradiction, assume the contrary. Let $M = \{k \leq p \mid x_k \in \operatorname{supp}(y)\}$. Then by assumption we have that for every $k \in M$, $\phi^{-1}(f)(x_k) = \phi^{-1}(g)(x_k)$. For every $k \in M$ let $I_k = \{i \leq n \mid x_k \notin K_i\}$, $J_k = \{i \leq m \mid x_k \notin L_i\}$ and

$$P_k = \bigcup_{i \in I_k} (K_i \cap X_k) \cup \bigcup_{i \in J_k} (L_i \cap X_k).$$

Then P_k is compact in T_k and $x_k \notin P_k$, so P_k is finite. Let $P = \bigcup_{k \in M} P_k$. Then P is finite and $P \cap \{x_k : k \in M\} = \emptyset$.

Define $f' : X \to \mathbb{R}$ by

$$f'(x) = \begin{cases} \phi^{-1}(f)(x) & \text{if } x \in P \cup \bigcup_{i \notin M} X_i, \\ \phi^{-1}(f)(x_k) & \text{if } x \in X_k \setminus P_k \text{ for } k \in M \end{cases}$$

and define $g' : X \to \mathbb{R}$ by

$$g'(x) = \begin{cases} \phi^{-1}(g)(x) & \text{if } x \in P \cup \bigcup_{i \notin M} X_i, \\ \phi^{-1}(g)(x_k) & \text{if } x \in X_k \setminus P_k \text{ for } k \in M. \end{cases}$$

Then f' and g' are continuous since for each $k \in M$, $X_k \setminus P_k$ is a neighborhood of x_k.

Let $U = \bigcup_{k \in M} X_k \setminus P_k \cup \operatorname{supp}(y)$. Then U is a neighborhood of $\operatorname{supp}(y)$ on which f' and g' coincide. Since ϕ is effective we have $\phi(f')(y) = \phi(g')(y)$.

On the other hand $f' \in \bigcap_{i=1}^{n} A(K_i, U_i)$. Indeed let $i \leq n$ and $x \in K_i$. If $x \in P \cup \bigcup_{i \notin M} X_i$, then $f'(x) = \phi^{-1}(f)(x) \in U_i$ since $\phi^{-1}(f) \in \bigcap_{i=1}^{n} A(K_i, U_i)$. If $x \notin P \cup \bigcup_{i \notin M} X_i$ we have $x \in X_k \setminus P_k$ for some $k \in M$. Since $x \in K_i \cap X_k$ and $x \notin P_k$ we have $x_k \in K_i$, so $f'(x) = \phi^{-1}(f)(x_k) \in U_i$. Similarly one can prove that

$g' \in \bigcap_{i=1}^{m} A(L_i, V_i)$. We then have $\phi(f') \in A(\{y\}, W_0)$ and $\phi(g') \in A(\{y\}, W_1)$. But this means $\phi(f')(y) \neq \phi(g')(y)$, which gives a contradiction. This completes the proof of the claim.

By lemma 4.1.11, there is a continuous linear function $\eta_1 : C_0(K) \to C_0(Y)$ such that for each $f \in C(K)$, $\eta_1(f) \mid K = f$ and there is a continuous linear function $\eta_2 : C_0(L) \to C_0(X)$ such that for each $f \in C(L)$, $\eta_2(f) \mid L = f$. Define

$\theta : C_0(K) \to C_0(L)$ by $\theta(f) = \phi^{-1}(\eta_1(f)) \mid L$, and
$\psi : C_0(L) \to C_0(K)$ by $\psi(g) = \phi(\eta_2(g)) \mid K$.

CLAIM 2: θ is a linear embedding.

It is easy to see that θ and θ are well-defined continuous functions. We claim that for every $h \in C_0(K)$ we have $\psi(\theta(h)) = h$. To the contrary suppose $\phi(\eta_2(\theta(h))) \mid K \neq \eta_1(h) \mid K$. By claim 1 we have $\eta_2(\theta(h)) \mid L \neq \phi^{-1}(\eta_1(h)) \mid L$. But this implies $\theta(h) \neq \theta(h)$. Contradiction. Hence θ is a linear embedding.

By claim 2, we have a linear embedding from $C_0([1, \omega^\omega])$ into $C_0([1, \alpha])$, where $\alpha < \omega^2$. However this contradicts theorem 2.4.1. This proves the theorem. □

§4.6. Partial results on ι_p^*-equivalence

We would like to have classification results for the spaces $C_p^*(X)$ as we had for the spaces $C_p(X)$ in the previous sections of this chapter. The theory developed there depends strongly on results derived in chapter 1 (corollary 1.2.15 (b)). Example 1.2.12 shows that the method for $C_p(X)$ cannot be used for $C_p^*(X)$, i.e., we cannot prove a theorem such as theorem 4.1.7 for "ι_p^*-equivalent pairs". We have to find another way to prove results for the function spaces $C_p^*(X)$.

In this section we will prove for ι_p^*-equivalent metric spaces X and Y, that $\kappa(X) < \omega$ if and only if $\kappa(Y) < \omega$. The proof of this result is a generalization of Pelant's proof that $C_p^*(T)$ and $C_p^*(\mathbb{Q})$ are not linearly homeomorphic (cf. [42]). The reader should compare this result with theorem 4.1.15, which states that for ι_p-equivalent metric spaces X and Y, $\kappa(X) \leq \omega$ if and only if $\kappa(Y) \leq \omega$.

We first need the following definition, which can be found in [23]. A family $\mathcal{F} \subset C(X)$ is ***equicontinuous*** if for every $x \in X$ and $\varepsilon > 0$, there is a neighborhood U of x in X such that for each $f \in \mathcal{F}$ and $y \in U$, $|f(x) - f(y)| < \varepsilon$. The following result is well-known.

4.6.1 PROPOSITION: *If $\mathcal{F} \subset C_u(X)$ is compact, then $\mathcal{F}$ is equicontinuous.*

PROOF: Let $x \in X$ and $\varepsilon > 0$. The family $\{<f, X, \varepsilon/3> : f \in \mathcal{F}\}$ is an open cover of $\mathcal{F}$. Since $\mathcal{F}$ is compact, there are $f_1, \ldots, f_n \in \mathcal{F}$ $(n \in \mathbb{N})$ such that $\{<f_i, X, \varepsilon/3> : i \leq n\}$ covers $\mathcal{F}$. Since each f_i is continuous, there is a neighborhood U of x such that for all $y \in U$ and for every $i \leq n$, $|f_i(y) - f_i(x)| < \varepsilon/3$. Now let $f \in \mathcal{F}$ and $y \in U$. There is $i \leq n$ such that $f \in <f_i, X, \varepsilon/3>$. This implies $|f_i(x) - f(x)| < \varepsilon/3$ and $|f_i(y) - f(y)| < \varepsilon/3$. Since $y \in U$, we now have

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \varepsilon. \square$$

4.6.2 THEOREM: *Let X and Y be first countable l_p^*-equivalent spaces. Then*

(a) $\kappa(X) < 1$ if and only if $\kappa(Y) < 1$,

(b) $\kappa(X) < 2$ if and only if $\kappa(Y) < 2$.

PROOF: For (a) observe that $\kappa(X) < 1$ if and only if $X = \emptyset$.

For (b) suppose $\kappa(X) < 2$ and $\kappa(Y) \geq 2$. Then by (a), $\kappa(X) = 1$, which gives that X is discrete. Since $\kappa(Y) \geq 2$ there is $y \in Y$ which is non-isolated. Let $\{U_n : n \in \mathbb{N}\}$ be a decreasing open base at y in Y. For every $n \in \mathbb{N}$ let f_n be a Urysohn function with $f_n(y) = 1$ and $f_n(Y \setminus U_n) = 0$. Then $f_n \to \chi_{\{y\}}$ pointwise in $\mathbb{R}^Y$. Since $\chi_{\{y\}} \notin C_p^*(Y)$, $\{f_n : n \in \mathbb{N}\}$ is closed and discrete in $C_p^*(Y)$.

Now let $\phi: C_p^*(X) \to C_p^*(Y)$ be a linear homeomorphism. Then by The Closed Graph Theorem, $\phi: C_u^*(X) \to C_u^*(Y)$ is also a linear homeomorphism. Since $C_u^*(X)$ and $C_u^*(Y)$ are Banach spaces, there is $k \in \mathbb{N}$ such that for every $f \in C^*(X)$ we have

$$\frac{1}{k}\|f\| \leq \|\phi(f)\| \leq k\|f\|.$$

Let $g_n = \phi^{-1}(f_n)$. Then $\|g_n\| \leq k\|f_n\| = k$. Hence $\{g_n : n \in \mathbb{N}\} \subset [-k, k]^X$. Since $[-k, k]^X$ is compact, $\{g_n : n \in \mathbb{N}\}$ has an accumulation point $g \in [-k, k]^X$. Since X is discrete $[-k, k]^X \subset C_p^*(X)$ and so $g \in C_p^*(X)$. However, since $\{f_n : n \in \mathbb{N}\}$ is closed and discrete in $C_p^*(Y)$, $\{g_n : n \in \mathbb{N}\}$ is closed and discrete in $C_p^*(X)$. Contradiction. $\square$

Before we prove our announced result we need two fairly simple lemmas. One deals with function spaces and the other one deals with nets.

4.6.3 LEMMA: *Let X be a metric space with $\kappa(X) < \omega$. There is a metric space Y such that $\kappa(Y) = \kappa(X)$ and $C_p^*(X) \sim C_{p,A}^*(Y)$ where $A = Y^{(1)}$.*

PROOF: We prove the lemma by induction on $\kappa(X)$. If $\kappa(X) = 1$, let $Y = X$. So suppose the lemma has been proved for metric spaces X with $\kappa(X) < n$ $(n > 1)$. Let X be a metric space with $\kappa(X) = n$ and let $B = X^{(1)}$. Then by proposition 2.3.2, the remark following lemma 2.3.6, and proposition 2.2.2 (a) $C_p^*(X) \sim C_p^*(B) \times C_{p,B}^*(X)$. Since $\kappa(B) = n - 1$ (corollary 2.2.3), there is by the inductive hypothesis a metric space Z such that $\kappa(Z) = \kappa(B)$ and $C_p^*(B) \sim C_{p,C}^*(Z)$ where $C = Z^{(1)}$. Then $C_p^*(X) \sim C_{p,C}^*(Z) \times C_{p,B}^*(X) = C_{p,B \cup C}^*(Z \oplus X)$. Let $Y = Z \oplus X$. Then $Y^{(1)} = B \cup C$ and $\kappa(Y) = \kappa(X)$. This finishes the proof of the lemma. □

4.6.4 LEMMA: *Let X be a space and B an infinite set. For every $b \in B$ let $f_b \in \mathbb{R}^X$ such that for every $x \in X$, $\{b \in B : f_b(x) \neq 0\}$ is finite. Furthermore let $\mathcal{S} = \{S \subset B : S$ is finite$\}$ and define a relation $\leq$ on $\mathcal{S}$ as follows: If $S_1, S_2 \in \mathcal{S}$ then $S_1 \leq S_2$ if $S_1 \subset S_2$. For every $S \in \mathcal{S}$ define $f_S = \Sigma_{b \in S} f_b$. Then $\{f_S : S \in \mathcal{S}\}$ is a net in $\mathbb{R}^X$ and $\lim_{S \in \mathcal{S}} f_S = \Sigma_{b \in B} f_b$.*

PROOF: It is easily seen that $\mathcal{S}$ is directed by $\leq$. Since every $S \in \mathcal{S}$ is finite, $f_S \in \mathbb{R}^X$, hence $\{f_S : S \in \mathcal{S}\}$ is a net in $\mathbb{R}^X$.

Now let $\varepsilon > 0$ and $P \subset X$ finite. For every $p \in P$ let $S_p = \{b \in B : f_b(p) \neq 0\}$ and $S_0 = \bigcup_{p \in P} S_p$. Then $S_0 \in \mathcal{S}$. Let $S \geq S_0$, $p \in P$ and $f = \Sigma_{b \in B} f_b$. Then

$$|f(p) - f_S(p)| = |\Sigma_{b \in B} f_b(p) - \Sigma_{b \in S} f_b(p)| = |\Sigma_{b \in S_p} f_b(p) - \Sigma_{b \in S_p} f_b(p)| = 0 < \varepsilon.$$

Hence $\lim_{S \in \mathcal{S}} f_S = f$. □

We now come to the result announced in the introduction of this section.

4.6.5 THEOREM: *Let X and Y be ℓ_p^*-equivalent metric spaces. Then $\kappa(X) < \omega$ if and only if $\kappa(Y) < \omega$.*

PROOF: Suppose $\kappa(X) < \omega$ and $\kappa(Y) \geq \omega$. By lemma 4.6.3 we may assume $C_{p,A}^*(X) \sim C_p^*(Y)$ where $A = X^{(1)}$. Let ϕ: $C_{p,A}^*(X) \to C_p^*(Y)$ be a linear homeomorphism. Then by The Closed Graph Theorem, ϕ: $C_{u,A}^*(X) \to C_u^*(Y)$ is also a linear homeomorphism ($C_{u,A}^*(X)$ has its obvious meaning). So there is $k \in \mathbb{N}$ such that for every $f \in C_{u,A}^*(X)$ we have

$$\frac{1}{k}\|f\| \leq \|\phi(f)\| \leq k\|f\|.$$

Let $B = X \setminus A$. Since every element of B is an isolated point in X, we have for each $x \in B$ that $f_x = \chi_{\{x\}} \in C^*_{p,A}(X)$, where $\chi_{\{x\}}$ is the characteristic function of x. Notice that for each $f \in C^*_{p,A}(X)$, $f = \Sigma_{x \in B} \alpha_x f_x$, where $\alpha_x = f(x)$. For each $x \in B$, let $g_x = \phi(f_x)$.

For every $y \in Y$, let $C_y = \{x \in B : g_x(y) \neq 0\}$. Since $C_y \subset \operatorname{supp}(y)$, C_y is finite for every $y \in Y$. Now define $b : Y \to \mathbb{R}$ by $b(y) = \Sigma_{x \in B} |g_x(y)|$. Notice that for every $y \in Y$, $b(y) = \Sigma_{x \in C_y} |g_x(y)|$, hence b is well-defined.

CLAIM 1: $\|b\| \leq 2k$.

For $y \in Y$, let $C_y^+ = \{x \in B : g_x(y) > 0\}$ and $C_y^- = \{x \in B : g_x(y) < 0\}$. Notice that $\|\Sigma_{x \in C_y^+} g_x\| = \|\phi(\Sigma_{x \in C_y^+} f_x)\| \leq k \cdot \|\Sigma_{x \in C_y^+} f_x\| = k$. Similarly we can prove that $\|\Sigma_{x \in C_y^-} g_x\| \leq k$. So

$$|b(y)| = |\Sigma_{x \in C_y^+} g_x(y) - \Sigma_{x \in C_y^-} g_x(y)| \leq |\Sigma_{x \in C_y^+} g_x(y)| + |\Sigma_{x \in C_y^-} g_x(y)| \leq 2k,$$

which proves the claim.

Now for $P \subset B$ finite let $\mathcal{M}_P = \{\Sigma_{x \in P} \alpha_x f_x : |\alpha_x| \leq k \text{ for } x \in P\}$. Notice that $\mathcal{M}_P = \Pi_{x \in P}[-k, k] \times \Pi_{x \in X \setminus P}\{0\}$.

CLAIM 2: For every $y \in Y$, $P \subset B$ finite and $\varepsilon > 0$, there is a neighborhood $U(y, P, \varepsilon)$ of y in Y such that for each $z \in U(y, P, \varepsilon)$ and $f \in \phi(\mathcal{M}_P)$, $|f(y) - f(z)| < \varepsilon$.

Notice that $\mathcal{M}_P$ is compact in $C^*_{p,A}(X)$. It is easily seen that for every $f \in \mathcal{M}_P$ and $\varepsilon > 0$, $B(f, \varepsilon) \cap \mathcal{M}_P = \langle f, P, \varepsilon \rangle \cap \mathcal{M}_P$. Since P is finite it now follows that $\mathcal{M}_P$ is compact in $C^*_{u,A}(X)$ and so $\phi(\mathcal{M}_P)$ is compact in $C^*_u(Y)$. Hence by proposition 4.6.1, $\phi(\mathcal{M}_P)$ is equicontinuous, from which the claim follows.

Now find $N \in \mathbb{N}$ such that $\frac{3}{4k}(N+1) \geq 2k$.

CLAIM 3: There are $y_0, \ldots, y_N \in Y$, $P_0, \ldots, P_N \subset B$ finite and $U_0, \ldots, U_N$ neighborhoods of respectively $y_0, \ldots, y_N$, such that

(1) for every $i \leq N$: $C_{y_i} \subset P_i$,
(2) $P_0 \subset P_1 \subset \cdots \subset P_N$,
(3) $U_0 \supset U_1 \supset \cdots \supset U_N$,
(4) for every $i \leq N$: $U_i \subset U(y_i, P_i, 1/4)$, and
(5) for every $i \leq N$: $y_i \in Y^{(N-i)}$.

We will prove this claim by induction. Since $\kappa(Y) \geq \omega$, we can find $y_0 \in Y^{(N)}$. Let $P_0 = C_{y_0}$ and $U_0 = U(y_0, P_0, 1/4)$. Suppose $y_0, \ldots, y_n, P_0, \ldots, P_n$ and $U_0, \ldots, U_n$ are found for $0 \leq n < N$. Since $y_n \in Y^{(N-n)}$ and $N - n \geq 1$, we can find $y_{n+1} \in U_n \setminus \{y_i : i \leq n\} \cap Y^{(N-(n+1))}$. Let $P_{n+1} = P_n \cup C_{y_{n+1}}$ and

$$U_{n+1} = U_n \cap U(y_{n+1}, P_{n+1}, \frac{1}{4}).$$

This completes the inductive construction and hence the proof of the claim.

Now let $g : Y \to [-1, 1]$ be a continuous function such that $g(y_i) = (-1)^i$ for $0 \leq i \leq N$. Then $\|g\| = 1$, so $\|\phi^{-1}(g)\| \leq k$. So $\phi^{-1}(g) = \Sigma_{x \in B} \alpha_x f_x$ with $|\alpha_x| \leq k$. Notice that $\Sigma_{x \in P_i} \alpha_x f_x \in \mathcal{M}_{P_i}$ for every $0 \leq i \leq N$.

CLAIM 4: $g = \Sigma_{x \in B} \alpha_x g_x$

Indeed, let $\mathcal{S} = \{S \subset B : S \text{ is finite}\}$ and for every $S \in \mathcal{S}$ let $f_S = \Sigma_{x \in S} \alpha_x f_x$. By lemma 4.6.4 $\phi^{-1}(g) = \lim_{S \in \mathcal{S}} f_S$ and $\Sigma_{x \in B} \alpha_x g_x = \lim_{S \in \mathcal{S}} \Sigma_{x \in S} \alpha_x g_x$. So

$$g = \phi(\phi^{-1}(g)) = \phi(\lim_{S \in \mathcal{S}} f_S) = \lim_{S \in \mathcal{S}} \phi(f_S) = \lim_{S \in \mathcal{S}} \Sigma_{x \in S} \alpha_x g_x = \Sigma_{x \in B} \alpha_x g_x,$$

and the claim is proved.

Let $0 \leq i \leq N$. Since $C_{y_i} \subset P_i$ (claim 3 (1)), we have by claim 4,

$$(-1)^i = \Sigma_{x \in B} \alpha_x g_x(y_i) = \Sigma_{x \in P_i} \alpha_x g_x(y_i).$$

By claim 3 (3) and (4), $y_N \in U(y_i, P_i, 1/4)$. Furthermore $\Sigma_{x \in P_i} \alpha_x g_x \in \phi(\mathcal{M}_{P_i})$, so by claim 2,

$$|\Sigma_{x \in P_i} \alpha_x g_x(y_N) - \Sigma_{x \in P_i} \alpha_x g_x(y_i)| < \frac{1}{4}.$$

If $i > 0$, we have by claim 3 (2)

$$\begin{aligned} |\Sigma_{x \in P_i \setminus P_{i-1}} \alpha_x g_x(y_N)| &= |\Sigma_{x \in P_i} \alpha_x g_x(y_N) - \Sigma_{x \in P_{i-1}} \alpha_x g_x(y_N)| \\ &= |\Sigma_{x \in P_i} \alpha_x g_x(y_N) - (-1)^i + (-1)^{i-1} - \Sigma_{x \in P_{i-1}} \alpha_x g_x(y_N) \pm 2| \\ &\geq 2 - |\Sigma_{x \in P_i} \alpha_x g_x(y_N) - \Sigma_{x \in P_i} \alpha_x g_x(y_i)| - \\ &\quad |\Sigma_{x \in P_{i-1}} \alpha_x g_x(y_N) - \Sigma_{x \in P_{i-1}} \alpha_x g_x(y_{i-1})| > \frac{3}{4}. \end{aligned}$$

If $i = 0$ and $P_{-1} = \emptyset$, then

$$|\Sigma_{x \in P_i \setminus P_{i-1}} \alpha_x g_x(y_N)| = |\Sigma_{x \in P_0} \alpha_x g_x(y_N) - \Sigma_{x \in P_0} \alpha_x g_x(y_0) + 1| > 1 - \frac{1}{4} = \frac{3}{4}.$$

So by claim 3 (2)

$$\Sigma_{x\in P_N}|\alpha_x g_x(y_N)| \geq \sum_{i=0}^{N}|\Sigma_{x\in P_i\setminus P_{i-1}}\alpha_x g_x(y_N)| > \frac{3}{4}(N+1),$$

hence

$$b(y_N)=\Sigma_{x\in P_N}|g_x(y_N)| \geq \Sigma_{x\in P_N}|\frac{\alpha_x}{k}g_x(y_N)| > \frac{3}{4k}(N+1)\geq 2k,$$

which gives a contradiction. □

The isomorphical classification of the function spaces in section 2.6 and the first part of lemma 2.3.7 are not valid for the function spaces of bounded continuous functions as is shown in the following

4.6.6 EXAMPLE: *There are l_p-equivalent countable metric locally compact spaces which are not l_p^*-equivalent.*

PROOF: Let $X=[1,\omega^2)$ and $Y=[1,\omega^\omega)$. Since X is an open subspace of $[1,\omega^2]$, $X^{(\alpha)}=X\cap[1,\omega^2]^{(\alpha)}$ (proposition 2.2.4). Hence by proposition 2.2.5 $\kappa(X)=2$. Similarly, $\kappa(Y)=\omega$, so by theorem 4.6.5, X and Y are not l_p^*-equivalent spaces. However, by example 2.6.18, X and Y are l_p-equivalent. □

4.6.7 REMARK: Let $2<n<\omega$. By proposition 2.4.4 there is $k\in\mathbb{N}$ such that

$$C_p^*([1,\omega])=C_p([1,\omega])\overset{k}{\sim}C_p([1,\omega^{n-1}])=C_p^*([1,\omega^{n-1}]).$$

So by lemma 2.3.7 (and the remark just before lemma 2.3.7),

$$C_p^*([1,\omega^2))=C_p^*(\bigoplus_{i=0}^{\infty}[1,\omega])\overset{k}{\sim}C_p^*(\bigoplus_{i=0}^{\infty}[1,\omega^n))=C_p^*([1,\omega^n)).$$

This implies that the following is not true: If X and Y are l_p^*-equivalent spaces and $n>2$, then $\kappa(X)<n$ if and only if $\kappa(Y)<n$.

Motivated by theorems 4.6.2 and 4.6.5 and remark 4.6.7 we state the following:

4.6.8 CONJECTURE: *Let X and Y be l_p^*-equivalent metric spaces and let α be a prime component. Then*

(a) $\kappa(X)<\alpha$ *if and only if* $\kappa(Y)<\alpha$, *and*

(b) $\kappa(X)<\alpha+1$ *if and only if* $\kappa(Y)<\alpha+1$.

References

[1] A.V. Arhangel'skiĭ, *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. 25 (1982) 852-855.

[2] A.V. Arhangelskiĭ, *A survey of C_p-theory*, Questions and Answers in General Topology 5 (1987) 1-109.

[3] J. Baars and J. de Groot, *An isomorphical classification of function spaces of zero-dimensional locally compact separable metric spaces*, Comm. Math. Univ. Carolinae 29 (1988) 577-595.

[4] J. Baars and J. de Groot, *On the l-equivalence of metric spaces*, Fund. Math. 137 (1991) 25-43.

[5] J. Baars, J. de Groot and J. van Mill, *A theorem on function spaces*, Proc. Amer. Math. Soc. 105 (1989) 1020-1024.

[6] J. Baars, J. de Groot, J. van Mill and J. Pelant, *On topological and linear homeomorphism of certain function spaces*, Top. and its Appl. 32 (1989) 267-277.

[7] J. Baars, J. de Groot and J. Pelant, *Function spaces on completely metrizable spaces*, to appear in Trans. Amer. Math. Soc..

[8] J.W. Baker, *Compact spaces homeomorphic to a ray of ordinals*, Fund. Math. 76 (1972) 19-27.

[9] J.W. Baker, *Ordinal subspaces of topological spaces*, Gen. Top. and its Appl. 3 (1973) 85-91.

[10] C. Bessaga and A. Pelczyński, *Spaces of continuous functions* IV *(on isomorphical classification of spaces of continuous functions)*, Studia Math. 19 (1960) 53-62.

[11] C. Bessaga and A. Pelczyński, *Selected topics in infinite-dimensional topology*, PWN Warszawa (1975).

[12] M. Bestvina, P.L. Bowers, J. Mogilski and J.J. Walsh, *Characterizing Hilbert space topology revisited*, Top. and its Appl. 24 (1986) 53-69.

[13] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional ab-*

solute retracts, Michigan Math. J. 33 (1986) 291-313

[14] K. Borsuk, *Theory of Retracts*, PWN Warszawa (1967).

[15] L.E.J. Brouwer, *On the structure of perfect sets of points*, Proc. Akad. Amsterdam, 12 (1910) 785-794.

[16] R. Cauty, *L'espace des fonctions continues d'un espace métrique dénombrable*, Proc. Amer. Math. Soc. 113 (1991) 493-501.

[17] T.A. Chapman, *Dense sigma-compact subsets of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. 154 (1971) 399-426.

[18] T.A. Chapman, *Lectures on Hilbert cube manifolds*, CMBS Regional Conf. Series in Math. No. 28, Amer. Math. Soc., Providence, R.I. (1976).

[19] J.J. Dijkstra, T. Grilliot, D.J. Lutzer and J. van Mill, *Function spaces of low Borel-complexity*, Proc. Amer. Math. Soc. 94 (1985) 703-710.

[20] T. Dobrowolski, S.P. Gulko and J. Mogilski, *Function spaces homeomorphic to the countable product of* l_f^2, Top. and its Appl. 34 (1990) 153-160.

[21] E.K. van Douwen, *The integers and topology*, Handbook of Set-theoretical topology, North-Holland Publishing Company, Amsterdam (1984) 111-167.

[22] J. Dugundji, *An extension of Tietze's theorem*, Pac. J. Math. 1 (1951) 353-367.

[23] J. Dugundji, *Topology*, Allyn and Bacon Inc, Boston (1966).

[24] R. Engelking, *General Topology*, PWN Warszawa (1977).

[25] L. Gillman and M. Jerison, *Rings of continuous functions*, Graduate texts in Mathematics, Springer Verlag, New York (1976).

[26] I. Gohberg and S. Goldberg, *Basic Operator Theory*, Birkhäuser (1981).

[27] S. Gulko, *The space* $C_p(X)$ *for countable infinite compact X is uniformly homeomorphic to* c_0, preprint.

[28] S.P. Gulko and A.V. Oskin, *Isomorphic classification of spaces of continuous functions on totally ordered compact sets*, Functional Analysis and its Applications 9 (1975) 56-57.

[29] J.A. Guthrie, *Ascoli theorems and the pseudocharacter of mapping spaces*, Bull. Austral. Math. Soc. 10, No.3 (1974) 403-408.

[30] D.W. Henderson, *Infinite dimensional manifolds are open subsets of Hilbert space*, Topology 9 (1970) 25-33.

[31] S.T. Hu, *Theory of Retracts*, Wayne State University Press, Detroit (1965).

[32] T. Jech, *Set Theory*, Academic Press New York, (1965).

[33] O.H. Keller, *Die Homoiomorphie der kompakten konvexen mengen in Hilbertschen raum*, Math. Ann. 105 (1931) 748-758.

[34] S.V. Kislyakov, *Classification of spaces of continuous functions of ordinals*, Siber. Math. J. 16 (1975) 226-231.

[35] K. Kuratowski and A. Mostowski, *Set theory*, North-Holland Publicing Company, Amsterdam (1976).

[36] S. Mazurkiewicz and W. Sierpiński, *Contributions à la topologie des ensembles dénomerables*, Fund. Math. 1 (1920) 17-27.

[37] R.A. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions*, Lecture Notes in Mathematics No. 1315, Springer Verlag, Berlin Heidelberg (1988).

[38] J. van Mill, *Topological equivalence of certain function spaces*, Comp. Math. 63 (1987) 159-188.

[39] J. van Mill, *Infinite Dimensional Topology: prerequisites and introduction*, North-Holland Publishing Company, Amsterdam (1989).

[40] J. Nagata, *On lattices of functions on topological spaces and functions on uniform spaces*, Osaka Math. J. 1, No. 2 (1949) 166-181.

[41] D. Pavlovskiĭ, *On spaces of continuous functions*, Soviet Math. Dokl. 22 (1980) 34-37.

[42] J. Pelant, *A remark on spaces of bounded continuous functions*, Indag. Math. 91 (1988) 335-338.

[43] A. Pelczyński and Z. Semadeni, *Spaces of continuous functions* III (*Spaces $C(\Omega)$ for Ω without perfect subsets*), Studia. Math. 18 (1959) 211-222.

[44] V.G. Pestov, *The coincidence of the dimension* dim *of l-equivalent topological spaces*, Soviet Math. Dokl. 26 (1982) 380-383.

[45] H.H. Schaeffer, *Topological Vector Spaces*, Macmillan series in advanced mathematics and theoretical physics, New York (1966).

[46] Z. Semadeni, *Banach spaces non-isomorphic to their Cartesian squares* II, Bull. Acad. Pol. Sci 8 (1960) 81-84

[47] Z. Semadeni, *Banach spaces of continuous functions*, PWN Warszawa (1971).

[48] W. Sierpiński, *Cardinal and ordinal numbers*, PWN Warszawa (1958).

[49] A.H. Stone, *Kernel constructions and Borel sets*, Trans. of the Am. Math. Soc. 107 (1963) 58-70.

[50] H. Toruńczyk, *On cartesian factors and the topological classification of linear metric spaces*, Fund. Math. 88 (1975) 71-86.

[51] H. Toruńczyk, *(G,K)-absorbing and skeletonized sets in metric spaces*, unpublished manuscript.

[52] V.V Uspenskiĭ, *On the topology of a free locally convex space*. Soviet Math. Dokl. 27 (1983) 781-785.

Special Symbols

$\mathbb{N}$	8	$\aleph_0$	8
$\mathbb{Q}$	8	$C(X)$	9
$\mathbb{R}$	8	$C^*(X)$	9
C	8	$<f, K, \varepsilon>$	9
$X \approx Y$	8	$C_p(X)$	10
$X \sim Y$	8	$C_0(X)$	10
$\bar{A}$	8	$C_b(X)$	10
$\mathrm{Int}\, A$	8	$A(K, U)$	11
$\min A$	8	$C_p^*(X)$	12
$\max A$	8	$C_0^*(X)$	12
$\inf A$	8	$C_b^*(X)$	12
$\sup A$	8	$C_u^*(X)$	12
$B(x, \varepsilon)$	8	$\mathrm{supp}\,(y)$	12
$\mathrm{diam}\, A$	8	$\mathrm{supp}\, A$	12
$d(X)$	8	ξ_x	24
$\lvert X \rvert$	8	$L(X)$	24
$\mathcal{P}(X)$	8	$L(f)$	26
$X \oplus Y$	8	$L^*(X)$	28
$\oplus_{i=1}^{\infty} X_i$	8	ℓ_p	31
$f \circ g$	8	ℓ_0	31
id_X, 1_X, 1	8	ℓ_b	31
$\pi_i: \Pi_{i \in I} X_i \to X_i$	8	ℓ_p^*	32
χ_A	8	ℓ_0^*	32
$\lVert . \rVert$	8	ℓ_b^*	32
$\mathrm{Ker}\, F$	8	$\mathrm{cof}\, \mathcal{K}(X)$	33
$\mathrm{span}\, \{v_1, \ldots, v_n\}$	8	$\bar{\alpha}$	45
$\mathrm{conv}\, \{v_1, \ldots, v_n\}$	8	$i(\phi)$	45
S^n	8	$cf(\lambda)$	46
B^{n+1}	8	ω_α	46
$W(\alpha)$	8	$\lim_{\xi<\alpha} \phi(\xi)$	46
$[1, \alpha]$	8	α'	49
$[1, \alpha)$	8	$X^{(\alpha)}$	51
ω	8	A^d	51
ω_1	8	$\kappa(X)$	54

X/A	54
$C_{p,A}(X)$	56
$C_{p,a}(X)$	56
$\Pi_{t\in T}E_t \overset{k}{\sim} \Pi_{s\in S}F_s$	56
$C_{p,\,0}([1,\alpha])$	61
$\Pi^*_{t\in T}E_t$	70
$\Pi_{\mathfrak{m}}E$	70
$\Pi^*_{t\in T}\Pi^*_{i\in I}E_{t,i}$	72
$C_{p,\,0}(X)$	99
$C^*_{p,\,0}(X)$	99
l^2	99
l^2_f	99
σ_ω	99
$C(X, Y)$	101
$\mathcal{H}(X, Y)$	101
$\mathcal{H}(X)$	101
$\hat{\mathrm{d}}(f, g)$	101
$\mathcal{I}(X)$	101
$\mathcal{I}_\sigma(X)$	101
Σ	103
Σ_n	103
$\ker \mathcal{A}$	105
ΣE	127
$W(X, x)$	130
$X^{\{\alpha\}}$	136
$X^{(\alpha,\beta)}$	138
$X^{\ll\alpha,\beta\gg}$	138
$X(\alpha, \beta)$	149
$X^{\ll\alpha,\beta,\gamma\gg}$	164
$X\ll\alpha, \beta, \gamma\gg$	164
$X^{<\alpha,\beta>}$	174
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14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

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38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

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47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

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